Gravitational Waves from 
Inspiralling Compact Binaries: 
PN Waveforms and Resummed Extensions

Bala R Iyer 
Raman Research Institute 
Bangalore, India

30 March 2002

Most Recent Review: 
Luc Blanchet, 
Living Reviews in Relativity, 
gr-qc 0202016 
Alessandra Buonanno, 
gr-qc 0203030
Phasing Formula

\[ t(v_F) = t_{LSO} + m \int_{v_F}^{v_{LSO}} dv \frac{E'(v)}{F(v)}, \]

\[ \phi(v_F) = \phi_{LSO} + 2 \int_{v_F}^{v_{LSO}} dv v^3 \frac{E'(v)}{F(v)}. \]

Newtonian Waveform

\[ a(t) = C_M(\pi M F(t))^{2/3}, \]

\[ \phi(t) = \phi_c - 2 \left[ \frac{(t_c - t)}{5M} \right]^{5/8}, \]

\[ \pi M F(t) = \left[ \frac{5M}{256(t_c - t)} \right]^{3/8}. \]

Invariant Velocity \( v : v^3 = \pi m F \)

Chirp Mass : \( \mathcal{M} = \nu^{3/5} m \)

Total mass : \( m \)

Symmetric mass ratio : \( \nu \)
GW from ICB

- Computations of GW from ICB requires control of 3 independent modules

1. Motion: Given a binary system, iterate EE to discuss Conservative motion of this system

2. Generation: Given the motion of a binary on a fixed orbit (circular), iterate EE to compute Multipoles of the Grav Field and thus GW flux

3. Radiation Reaction: Given the flux of radiated energy (AM), use balance arguments to compute the effect on the orbit
MOTION

• Status of PN EOM *satisfactory*
  Agreement between different approaches and techniques

• 2.5PN

  Damour, Deruelle: Harmonic coords, Riesz regularisation

  Schafer: ADM, Hadamard partie finie

  Kopeijkin and Grischuk: Physical computation using self gravitating extended bodies

  Blanchet, Faye and Ponsot: Direct PN iteration, Matching

  Itoh, Futamase and Asada: Variant of surface integral approach of EIH
• 3PN

Jaranowski, Schafer and Damour:  
ADM coords, Hadamard regularisation,  
EOM has an arbitrary parameter $\omega_{static}$

Blanchet, Faye and Andrade:  
Harmonic coords, (extended) Hadamard regularisation, EOM has an arbitrary parameter $\lambda$

$$\lambda = -\frac{3}{11}\omega_{static} - \frac{1987}{3080}$$

• The undetermined constant reflects the incompleteness of the Hadamard regularisation

• Hadamard regularisation does not satisfy distributivity of products $(FG)_1 \neq (F)_1(G)_1$. Violates Leibniz rule for differentiation of a product
• Dimensional Regularisation preserves the gauge symmetry of perturbative GR underlying the link between Bianchi identities and EOM and hence respects ALL basic properties of algebraic and differential calculus of ordinary functions

• Damour, Jaranowski and Schafer: Dimensional regularisation gives $\omega_{static} = 0$ so that $\lambda = -\frac{1987}{3080} = -0.645..$

• 3PN EOM and ALL conserved quantities available for General Orbits
• Computation without regularisation:
  Calculate 3PN EOM for extended bodies taking into account internal structure (pressure, density..) and then take its limit as ‘radius’ goes to zero. Compare with the point mass regularised result

  2PN: Kopeijkin and Grischuk implemented this and showed *effacement* of internal structure

  3PN: Can one determine $\lambda$???
  Is this consistent with $\omega_{\text{static}} = 0$
  (Blanchet, Esposito-Farese, Poujade)

• Can one compute EOM in harmonic coords using dimensional regularisation and determine $\lambda$?
• Upto now one has been discussing the conservative motion of the binary

• The radiative part of the EOM is available only upto leading order (2.5PN)

• Deriving the full *relative* 3PN/3.5PN radiation reaction ie *absolute* 5.5PN/6PN contributions is impossible with present technology

• Thus we move to the second module
Apply wave generation formula to compute the work done by radiation reaction force i.e. total energy flux at null infinity

Computation of Source multipole moments $I_L$ and $J_L$

Determination and control of Tails and non-linear effects relating source moments to Radiative moments

2PN
Blanchet, Damour, BRI, Will and Wiseman
BDI - Multipolar Post Minkowskian method, Hadamard/Riesz self-field regularisation
WW - Direct Integration of Relaxed Einstein (DIRE) ; Epstein-Wagoner-Thorne + retarded integral
• Mathematical Equivalence of Both approaches (Blanchet)

• 3PN Instantaneous part
  Blanchet, BRI, Joguet; Circular Orbits;
  Harmonic coords + Hadamard regularn of infinite self-field
  General orbits (In Progress: Blanchet, BRI)

• Hereditary part: Blanchet
  Tails: 1.5PN, 2.5PN, 3.5PN
  Tails of Tails, (Tail)$^2$: 3PN

• 3 undetermined constants in the the Mass Quadrupole combine to a single undetermined constant $\theta$ in GW
  Luminosity in addition to the $\lambda$ coming from EOM
DETERMINATION of $\theta$??

- Can $\theta$ be computed by an Extended Body Computation? (Blanchet, BRI..)

- Can we formulate the wave generation in ADM coords? 2PN, Tails, ????

- Can self-fields in harmonic coordinates be controlled by the Dimensional Regularn in the generation problem??
  Need to first discuss EOM in Harmonic coordinates with Dimensional Regularn
  Setting up the entire MPM generation formalism in $d$ dimensions seems non-trivial
  Rotation group in higher dimensions, Propagator in higher dimensions,
  Backscattering/Tails..
  Can one be smart enough to apply Dimensional regularisation only where required without setting up the whole edifice???
\[ h_{+,x} = \frac{2Gm\eta}{c_0^2 r} \left( \frac{G\omega}{c_0^3} \right)^{2/3} \left\{ H_{+,x}^{(0)} + x^{1/2}H_{+,x}^{(1/2)} + xH_{+,x}^{(1)} + x^{3/2}H_{+,x}^{(3/2)} + x^2H_{+,x}^{(2)} \right\} \]

\[ H_{+}^{(0)} = -(1 + c^2) \cos 2\psi , \]

\[ H_{+}^{(1/2)} = -\frac{s \delta m}{8 m} \left[ (5 + c^2) \cos \psi - 9(1 + c^2) \cos 3\psi \right] , \]

\[ H_{+}^{(1)} = \frac{1}{6} \left[ (19 + 9c^2 - 2c^4) - \eta(19 - 11c^2 - 6c^4) \right] \cos 2\psi \]

\[ - \frac{4}{3} s^2 (1 + c^2)(1 - 3\eta) \cos 4\psi , \]

\[ H_{+}^{(3/2)} = \frac{s}{192 m} \delta m \left\{ \left[ (57 + 60c^2 - 4c^4) - 2\eta(49 - 12c^2 - 4c^4) \right] \cos \psi \right. \]

\[ - \frac{27}{2} \left[ (73 + 40c^2 - 9c^4) - 2\eta(25 - 8c^2 - 9c^4) \right] \cos 3\psi \]

\[ + \frac{625}{2} (1 - 2\eta)s^2 (1 + c^2) \cos 5\psi \left\} - 2\pi (1 + c^2) \cos 2\psi , \right. \]

\[ H_{+}^{(2)} = \frac{1}{120} \left[ (22 + 396c^2 + 145c^4 - 5c^6) + \right. \]

\[ + \frac{5}{3} \eta(706 - 216c^2 - 251c^4 + 15c^6) \]

\[ - 5\eta^2(98 - 108c^2 + 7c^4 + 5c^6) \right] \cos 2\psi \]

\[ + \frac{2}{15} s^2 \left[ (59 + 35c^2 - 8c^4) - \frac{5}{3} \eta(131 + 59c^2 - 24c^4) \right. \]

\[ + 5\eta^2(21 - 3c^2 - 8c^4) \right] \cos 4\psi \]

\[ - \frac{81}{40} (1 - 5\eta + 5\eta^2)s^4 (1 + c^2) \cos 6\psi \]

\[ + \frac{s}{40 m} \delta m \left\{ \left[ 11 + 7c^2 + 10(5 + c^2) \ln 2 \right] \sin \psi - 5\pi (5 + c^2) \cos \psi \right. \]

\[ - 27 \left[ 7 - 10 \ln(3/2) \right] (1 + c^2) \sin 3\psi + 135\pi (1 + c^2) \cos 3\psi \right\} , \]

\[ \psi = \phi - \frac{2Gm\omega}{c_0^3} \ln \left( \frac{\omega}{\omega_0} \right) , \]
\[ \Theta = \frac{c_0^3 \eta}{5Gm} (t_c - t) , \]

\[ \phi(t) = \phi_c - \frac{1}{\eta} \left\{ \Theta^{5/8} + \left( \frac{3715}{8064} + \frac{55}{96} \eta \right) \Theta^{3/8} - \frac{3\pi}{4} \Theta^{1/4} \right\} + \left( \frac{9275495}{14450688} + \frac{284875}{258048} \eta + \frac{1855}{2048} \eta^2 \right) \Theta^{1/8} \right\} , \]

\[ \omega(t) = \frac{c_0^3}{8Gm} \left\{ \Theta^{-3/8} + \left( \frac{743}{2688} + \frac{11}{32} \eta \right) \Theta^{-5/8} - \frac{3\pi}{10} \Theta^{-3/4} \right\} + \left( \frac{1855099}{14450688} + \frac{56975}{258048} \eta \right) \Theta^{-7/8} \right\} \]

Blanchet, Damour, BRI, Will, Wiseman
\[ E = \frac{-\mu c^2 \gamma}{2} \left\{ 1 + \left( -\frac{7}{4} + \frac{1}{4 \nu} \right) \gamma + \right. \\
+ \left( -\frac{7}{8} \nu + \frac{1}{8 \nu^2} \right) \gamma^2 + \\
+ \left( -\frac{235}{64} \nu + \\
+ \left[ \frac{106301}{6720} - \frac{123}{64} \pi^2 + \frac{22}{3} \ln\left( \frac{r}{r_0'} \right) - \frac{22}{3} \frac{\lambda}{\nu} + \\
+ \frac{27}{32} \nu^2 + \frac{5}{64} \nu^3 \right] \gamma^3 \right\}. \]

\[ E = \frac{-\mu c^2 x}{2} \left\{ 1 + \left( -\frac{3}{4} - \frac{1}{12 \nu} \right) x + \\
+ \left( -\frac{27}{8} \nu + \frac{19}{8} \nu^2 + \frac{1}{24} \nu^2 \right) x^2 + \\
+ \left( -\frac{675}{64} \nu^2 + \left[ \frac{209323}{4032} - \frac{205}{96} \pi^2 - \frac{110}{9} \frac{\lambda}{\nu} \right] x^3 - \\
- \frac{155}{96} \nu^2 - \frac{35}{5184} \nu^3 \right\}. \]

\[ E_{\text{test}} = \mu c^2 \left[ (1 - 2x)(1 - 3x)^{-1/2} - 1 \right] \]

Blanchet, Faye, (Andrade)
\[
\mathcal{L} = \frac{32c^5}{5G} \gamma \nu^2 \left\{ 1 + \left( -\frac{2927}{336} - \frac{5}{4} \nu \right) \gamma + 4\pi \gamma^{3/2} + \left( \frac{293383}{9072} + \frac{380}{9} \nu \right) \gamma^2 \right. \\
+ \left( -\frac{25663}{672} - \frac{109}{8} \nu \right) \pi \gamma^{5/2} \\
+ \left( \frac{129386791}{7761600} + \frac{16\pi^2}{3} - \frac{1712}{105} C - \frac{856}{105} \ln(16\gamma) \right. \\
+ \left[ -\frac{332051}{720} + \frac{110}{3} \ln \left( \frac{r}{r'_{0}} \right) + \frac{123\pi^2}{64} + 44\lambda - \frac{88}{3} \theta \right] \nu - \frac{383}{9} \nu^2 \right) \gamma^3 \\
+ \left( \frac{90205}{576} + \frac{3772673}{12096} \nu + \frac{32147}{3024} \nu^2 \right) \pi \gamma^{7/2} + O(\gamma^4) \right\}
\]

\[
\mathcal{L} = \frac{32c^5}{5G} x^5 \nu^2 \left\{ 1 + \left( -\frac{1247}{336} - \frac{35}{12} \nu \right) x + 4\pi x^{3/2} \right. \\
+ \left( -\frac{44711}{9072} + \frac{9271}{504} \nu + \frac{65}{18} \nu^2 \right) x^2 + \left( -\frac{8191}{672} - \frac{535}{24} \nu \right) \pi x^{5/2} \\
+ \left( \frac{6643739519}{69854400} + \frac{16\pi^2}{3} - \frac{1712}{105} C - \frac{856}{105} \ln(16x) \right. \\
+ \left[ -\frac{11497453}{272160} + \frac{41\pi^2}{48} + \frac{176}{9} \lambda - \frac{88}{3} \theta \right] \nu - \frac{94403}{3024} \nu^2 - \frac{775}{324} \nu^3 \right) x^3 \\
+ \left. \left( -\frac{16285}{504} + \frac{176419}{1512} \nu + \frac{19897}{378} \nu^2 \right) \pi x^{7/2} + O(x^4) \right\}
\]

Blanchet, BRI, Joguet
NONLINEAR EFFECTS

• The 1.5PN term is the Tail term. In addition to the (instantaneous) Linear wave emitted at retarded time, we have the (hereditary) Tail wave emitted in the past and scattered off the static (Schwarzschild) gravitational field. This is important for compact binaries.

• The 2.5PN term is the Nonlinear Memory term. It is the gravitational radiation from the linear gravitations. It has poor observational consequences.

• Upto this order the Instantaneous and Hereditary terms remain disjoint. No longer true at 3PN order.

• At 3PN we have the Tail of Tails and the (Tail)$^2$ terms. This is important for CB.
ECCENTRIC BINARIES

- Will and Wiseman, Gopakumar and BRI
  2PN Energy Flux, Waveform

- Gopakumar and BRI
  AM Flux, Evolution of orbital elements,
  GW polarisations without inspiral but 2PN
  accurate periastron precession

- GW polarisations with RR
  (In progress: Damour, Gopakumar, BRI)
RADIATION REACTION

• We assume an energy balance equation

\[ \frac{dE}{dt} = \mathcal{L} \]

• Though *physically obvious*, no general proof from first principles of GR of the correctness of the above balance eqn beyond 1PN/1.5PN

• Blanchet, Faye, BRI, Joguet
  3.5PN GW Phasing
Contributions to the accumulated number $N = \frac{1}{\pi}(\phi_{\text{ISCO}} - \phi_{\text{seismic}})$ of gravitational-wave cycles. Frequency entering the bandwidth is $f_{\text{seismic}} = 10$ Hz; terminal frequency is assumed to be at the Schwarzschild innermost stable circular orbit $f_{\text{ISCO}} = \frac{c^3}{6^{3/2}\pi Gm}$. The 3PN term depends on the unknown parameter $\hat{\theta} = \theta - \frac{7}{3}\lambda$ (we have $\hat{\theta} = \theta + \frac{1987}{1320}$ using the value of $\lambda$ following from $\omega_{\text{static}} = 0$).

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>Newtonian</td>
<td>16031</td>
<td>3576</td>
<td>602</td>
</tr>
<tr>
<td>1PN</td>
<td>441</td>
<td>213</td>
<td>59</td>
</tr>
<tr>
<td>1.5PN</td>
<td>−211</td>
<td>−181</td>
<td>−51</td>
</tr>
<tr>
<td>2PN</td>
<td>9.9</td>
<td>9.8</td>
<td>4.1</td>
</tr>
<tr>
<td>2.5PN</td>
<td>−12.2</td>
<td>−20.4</td>
<td>−7.5</td>
</tr>
<tr>
<td>3PN</td>
<td>2.5+0.5 $\hat{\theta}$</td>
<td>2.2+0.4 $\hat{\theta}$</td>
<td>2.1+0.4 $\hat{\theta}$</td>
</tr>
<tr>
<td>3.5PN</td>
<td>−1.0</td>
<td>−1.9</td>
<td>−0.9</td>
</tr>
</tbody>
</table>

$A \equiv 2 \times 1.4M_{\odot}$  
$B \equiv 10M_{\odot} + 1.4M_{\odot}$  
$C \equiv 2 \times 10M_{\odot}$
Why go beyond Taylor approximants

- Standard PN expansion is very slowly and poorly convergent

- The convergence may be improved by Resummation methods like Padé approxts

- Effective one body method is a very efficient way to investigate the conservative motion of the binary

- The early inspiral is well modelled by the adiabatic approximation

- Need to go beyond adiabatic approximation since the PN parameter is not small near the LSO
• $E, e, j$, methods to locate LSO cannot be used beyond the adiabatic approximation

• A combination of Resummation methods and EOB is necessary to go beyond the adiabatic approximation and extend the validity of PN expansions. Allows one to discuss late inspiral, plunge and subsequent merger

• Buonanno - Damour: Transition between inspiral and plunge gradual. Location of LSO unimportant. Precise evaluation of RR more important. EOB least sensitive to 3PN coefficients.
• EOB condenses essential information of dynamics in one fn: the radial potential

\[ A(r) = 1 - (2)u + (2\nu)u^3 + a_4(\nu)u^4 + \cdots \]

\[ u \sim \frac{Gm}{r} \]

• Most of the gauge related complications of 2 body EOM get absorbed in the mapping between 2 body \(\rightarrow\) EOB

• Mapping preserves adiabatic invariants

• GWDA using 3PN EOB and sensitivity of the overlaps to flexibility parameters and 3PN unknown parameters (Damour, BRI, Jaranowski and Sathyaprakash - In Progress)

• Though 3PN non-resummed good upto LSO, it may not be good enough to discuss plunge to coalescense
• In spinning case, a ‘deformed Kerr’ captures all SO and most SS

• Applies to arbitrary $S_1$ and $S_2$ orientation

• Also in the spinning case Non-resummed results not as good as the EOB

• Spinning EOB implies that spin effects are small since there is a cancellation of energy increase by spin KE by energy decrease due to SO