Toward Astrophysical Black-Hole Binaries

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Mar. 29, 2002

Abstract
A formalism for constructing initial data representing black-hole binaries in quasi-equilibrium is developed. If each black hole is assumed to be in quasi-equilibrium, then a complete set of boundary conditions for all initial data variables can be developed. This formalism should allow for the construction of completely general quasi-equilibrium black hole binary initial data.


Collaborators: Harald Pfeiffer & Saul Teukolsky (Cornell)
Motivation

• How do we go about constructing improved initial-data sets that more accurately represent astrophysical compact binary systems?

• How do we define astrophysically realistic data?

Focus Issues

• Which decomposition of the constraints will be used?

• How do we choose boundary conditions so that the constraints are well-posed and yield solutions with the desired physical content?

• What choices for the spatial and temporal gauge are compatible with the desired physical content?

• How do we fix the remaining freely specifiable data so as to yield the desired physical content?
The 3 + 1 Decomposition

Lapse: $\alpha$

Spatial metric: $\gamma_{ij}$

Shift vector: $\beta^i$

Extrinsic Curvature: $K_{ij}$

Time vector: $t^\mu = \alpha n^\mu + \beta^\mu$

$$d s^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$$

$$\gamma_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$$

$K_{\mu\nu} = -\frac{1}{2} \gamma^\alpha_{\mu} \gamma^\beta_{\nu} \partial_i g_{\alpha\beta}$

**Constraint equations**

$$\bar{R} + K^2 - K_{ij} K^{ij} = 16\pi \rho$$

$$\bar{\nabla}_j \left( K^{ij} - \gamma^{ij} K \right) = 8\pi j^i$$

where:

- $S_{\mu\nu} = \gamma^\alpha_{\mu} \gamma^\beta_{\nu} T_{\alpha\beta}$
- $j_\mu = -\gamma^\nu_{\mu} n_\alpha T_{\nu\alpha}$
- $\rho = n^\mu n_\nu T_{\mu\nu}$
- $T_{\mu\nu} = S_{\mu\nu} + 2 n_{(\mu} j_{\nu)} + n_\mu n_\nu \rho$

**Evolution equations**

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + \bar{\nabla}_i \beta_j + \bar{\nabla}_j \beta_i$$

$$\partial_t K_{ij} = -\bar{\nabla}_i \bar{\nabla}_j \alpha + \alpha \left[ \bar{R}_{ij} - 2 K_{i\ell} K^\ell_j + KK_{ij} - 8\pi S_{ij} + 4\pi \gamma_{ij} (S - \rho) \right]$$

$$+ \beta^\ell \bar{\nabla}_\ell K_{ij} + K_{i\ell} \bar{\nabla}_j \beta^\ell + K_{j\ell} \bar{\nabla}_i \beta^\ell$$

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“Traditional” Black-Hole Data

Conformal flatness and maximal slicing

\[ \tilde{\gamma}_{ij} = f_{ij} \ (\text{flat}) \]
\[ K = 0 \]

\[ \Rightarrow \left\{ \begin{align*}
\Delta_{\mathbf{L}} X^i &= 0 \\
\end{align*} \right. \Rightarrow \]
\[ \tilde{\Delta}^2 \psi + \frac{1}{8} \psi^{-7} \tilde{A}_{ij} \tilde{A}^{ij} = 0 \]

Bowen-York solution

Analytic solutions for \( \tilde{A}^{ij} \)
(conformal tracefree extrinsic curvature)

Three general solution schemes

Conformal Imaging-[6]

Inversion symmetry
inner-BC

Apparent Horizon BC-[11]

Apparent horizon inner-BC

Puncture Method-[4]

No inner-BC: singular behavior factored out

All methods can produce very general configurations of multiple black holes, but are fundamentally limited by choices for \( \tilde{\gamma}_{ij} \) and Bowen-York \( \tilde{A}^{ij} \).
“Better” Black-Hole Data

What is wrong with “traditional” BH initial data?

- Results disagree with PN predictions for black holes in quasi-circular orbits.
- There is no control of the initial “wave” content.
- Spinning holes are not represented well.

How do we construct improved BH initial data?

We must carefully choose the

- initial dynamical degrees of freedom [in $\tilde{\gamma}_{ij}$ and $\tilde{A}^{ij}_{TT}$]
- initial temporal and spatial gauge degrees of freedom [in $\tilde{\gamma}_{ij}$ and $K$]
- boundary conditions on the constrained degrees of freedom [in $\psi$ and $X^i$]

so as to conform to the desired physical content of the initial data.

- For black holes in quasi-circular orbits, we can use the principle of quasi-equilibrium to guide our choices.
- Quasi-equilibrium is a dynamical concept and we can simplify our task by choosing a decomposition of the initial-data variables that has connections to dynamics.
Conformal Thin-Sandwich Decomposition[13]

\begin{align*}
\gamma_{ij} &= \psi^4 \tilde{\gamma}_{ij} \\
K^{ij} &= \frac{\psi^{-10}}{2\tilde{\alpha}} \left[ (\tilde{L}_\beta)^{ij} - \tilde{u}^{ij} \right] + \frac{1}{3} \gamma^{ij} K \left\{ \begin{array}{l}
\tilde{u}_{ij} \equiv \partial_t \tilde{\gamma}_{ij} \ (\tilde{u}^i_i = 0) \\
\tilde{\alpha} \equiv \psi^{-6} \alpha
\end{array} \right.
\end{align*}

Hamiltonian Const. \quad \tilde{\nabla}^2 \psi - \frac{1}{8} \psi \tilde{R} - \frac{1}{12} \psi^5 K^2 + \frac{1}{8} \psi^{-7} \tilde{A}_{ij} \tilde{A}^{ij} = -2\pi \psi^5 \rho

Momentum Const. \quad \tilde{\Delta}_\beta^i - (\tilde{L}_\beta)^{ij} \tilde{\nabla}_j \tilde{\alpha} = \frac{4}{3} \tilde{\alpha} \psi^6 \tilde{\nabla}^i K + \tilde{\alpha} \tilde{\nabla}_j \left( \frac{1}{\tilde{\alpha}} \tilde{u}^{ij} \right) + 16\pi \tilde{\alpha} \psi^{10} j^i

\tilde{A}^{ij} \equiv \frac{1}{2\tilde{\alpha}} \left[ (\tilde{L}_\beta)^{ij} - \tilde{u}^{ij} \right]

Constrained vars : \psi and \beta^i
Freely specified : \tilde{\gamma}_{ij}, \tilde{u}^{ij}, K, and \tilde{\alpha}

\tilde{u}^{ij} and \beta^i have a simple physical interpretation, unlike \tilde{A}^{ij}_{TT} and X^i.

Quasi-equilibrium \Rightarrow \left\{ \begin{array}{l}
\tilde{u}^{ij} = 0 \\
\partial_t K = 0 \ (\text{Const. on } \alpha)
\end{array} \right.

Constr. Tr(K) eqn. \quad \tilde{\nabla}^2 (\alpha \psi) - \alpha \left[ \frac{1}{8} \psi \tilde{R} + \frac{5}{12} \psi^5 K^2 + \frac{7}{8} \psi^{-7} \tilde{A}_{ij} \tilde{A}^{ij} \\
+ 2\pi \psi^5 K (\rho + 2S) \right] = \psi^5 \beta^i \tilde{\nabla}_i K

\text{– Greg Cook – (WFU Physics)}
Equations of Quasi-Equilibrium

\[ \text{Ham. \& Mom. const. eqns. from Conf. TS} \]
\[ \text{+ Const. } \text{Tr}(K) \text{ eqn.} \]
\[ \Rightarrow \text{Eqns. of Quasi-Equilibrium} \]

With \( \tilde{\gamma}_{ij} = f_{ij}, \tilde{u}^{ij} = 0, \) and \( K = 0, \) these equations have been widely used to construct binary neutron star initial data \([1, 10, 2, 12]\).

Binary neutron star initial data require:

- boundary conditions at infinity compatible with asymptotic flatness and corotation.
  \[ \psi|_{r \to \infty} = 1, \quad \beta^{i}|_{r \to \infty} = \Omega \left( \frac{\partial}{\partial \phi} \right)^{i}, \quad \alpha|_{r \to \infty} = 1 \]

- compatible solution of the equations of hydrostatic equilibrium. \((\Rightarrow \Omega)\)

Binary black hole initial data require:

- a means for choosing the angular velocity of the orbit \( \Omega. \)

\* with excision, inner boundary conditions are needed for \( \psi, \beta^{i}, \) and \( \tilde{\alpha}. \)

Gourgoulhon, Grandclément, & Bonazzola \([8, 9]\): Black-hole binaries with \( \tilde{\gamma}_{ij} = f_{ij}, \tilde{u}^{ij} = 0, K = 0, \) “inversion-symmetry”, and “Killing-horizon” conditions on the excision boundaries.

“Solutions” require constraint violating regularity condition imposed on inner boundaries!
Constructing Regular Binary Black Hole QE ID

Why does the GGB approach have problems?

• Inversion-symmetry demands $\tilde{\alpha} = 0$ & $K = 0$ on the inner boundary.

• It is hard to move beyond $\tilde{\gamma}_{ij} = f_{ij}$.

$\tilde{A}^{ij} \equiv \frac{1}{2\tilde{\alpha}} \left[ (\tilde{L}_\beta)^{ij} - \tilde{u}^{ij} \right]$

How do we proceed?

• Find a method that allows for general choices of $\tilde{\gamma}_{ij}$ & $K$.

☆ Eliminate dependence on inversion symmetry by letting the physical condition of quasi-equilibrium dictate the boundary conditions.

Approach

• Demand that the excision (inner) boundary be an apparent horizon.

• Demand that the apparent horizon be in quasi-equilibrium.
The Inner Boundary

Extrinsic curvature of $S$ embedded in $\Sigma$

$$H_{ij} \equiv -\frac{1}{2} h_i^k h_j^\ell \mathcal{L} s_{k\ell}$$

Projections of $K_{ij}$ onto $S$

$$J_{ij} \equiv h_i^k h_j^\ell K_{k\ell}$$

$$J_i \equiv h_i^k s^\ell K_{k\ell}$$

$$J \equiv h^{ij} J_{ij} = h^{ij} K_{ij}$$

Expansion of null rays

$$\theta \equiv h^{ij} \Sigma_{ij} = \frac{1}{\sqrt{2}} (J + H)$$

$$\dot{\theta} \equiv h^{ij} \dot{\Sigma}_{ij} = \frac{1}{\sqrt{2}} (J - H)$$

Shear of null rays

$$\sigma_{ij} \equiv \Sigma_{ij} - \frac{1}{2} h_{ij} \theta$$

$$\dot{\sigma}_{ij} \equiv \dot{\Sigma}_{ij} - \frac{1}{2} h_{ij} \dot{\theta}$$

$$s_i \equiv \frac{\tilde{\nabla}_i \tau}{|\tilde{\nabla} \tau|}$$

$$h_{ij} \equiv \gamma_{ij} - s_i s_j$$

$$k^\mu \equiv \frac{1}{\sqrt{2}} (n^\mu + s^\mu)$$

$$\dot{k}^\mu \equiv \frac{1}{\sqrt{2}} (n^\mu - s^\mu)$$
AH and QE Conditions on the Inner Boundary

The quasi-equilibrium inner boundary conditions start with the following assumptions:

1. The inner boundary $\mathcal{S}$ is a (MOTS):
   \[ \theta = 0 \]

2. The inner boundary $\mathcal{S}$ doesn’t move:
   \[ \mathcal{L}_\zeta \tau = 0 \text{ and } \hat{\nabla}_i \mathcal{L}_\zeta \tau \equiv h^j_i \nabla_j \mathcal{L}_\zeta \tau = 0 \]
   \[ t^\mu = \alpha n^\mu + \beta^\mu \]
   \[ \zeta^\mu \equiv \alpha n^\mu + \beta_\perp s^\mu \]
   \[ \beta_\perp \equiv \beta^i s_i \]

3. The inner boundary $\mathcal{S}$ remains a MOTS\cite{7}:
   \[ \mathcal{L}_\zeta \theta = 0 \text{ and } \mathcal{L}_\zeta \dot\theta = 0 \]

4. The horizons are in quasi-equilibrium:
   \[ \sigma_{ij} = 0 \text{ and no matter is on } \mathcal{S} \]
Evolution of the Expansions

\[ \mathcal{L}_\zeta \theta = \frac{1}{\sqrt{2}} \left[ \theta (\theta + \frac{1}{2} \dot{\theta} - \frac{1}{\sqrt{2}} K) + \mathcal{E} \right] (\beta_\perp + \alpha) \]

\[ + \frac{1}{\sqrt{2}} \left[ \theta (\frac{1}{2} \theta - \frac{1}{2} \dot{\theta} - \frac{1}{\sqrt{2}} K) + \mathcal{D} + 8\pi T_{\mu\nu} k^\mu k^\nu \right] (\beta_\perp - \alpha) \]

\[ + \theta s^i \bar{\nabla}_i \alpha, \]

\[ \mathcal{L}_\zeta \dot{\theta} = \frac{1}{\sqrt{2}} \left[ \dot{\theta} (\dot{\theta} + \frac{1}{2} \theta - \frac{1}{\sqrt{2}} K) + \dot{\mathcal{E}} \right] (\beta_\perp - \alpha) \]

\[ - \frac{1}{\sqrt{2}} \left[ \dot{\theta} (\frac{1}{2} \dot{\theta} - \frac{1}{2} \theta - \frac{1}{\sqrt{2}} K) + \dot{\mathcal{D}} + 8\pi T_{\mu\nu} k^\mu k^\nu \right] (\beta_\perp + \alpha) \]

\[ - \dot{\theta} s^i \bar{\nabla}_i \alpha, \]

\[ \mathcal{D} \equiv h^{ij} (\bar{\nabla}_i + J_i)(\bar{\nabla}_j + J_j) - \frac{1}{2} R \]

\[ \dot{\mathcal{D}} \equiv h^{ij} (\bar{\nabla}_i - J_i)(\bar{\nabla}_j - J_j) - \frac{1}{2} R \]

\[ \mathcal{E} \equiv \sigma_{ij} \sigma^{ij} + 8\pi T_{\mu\nu} k^\mu k^\nu \]

\[ \dot{\mathcal{E}} \equiv \dot{\sigma}_{ij} \sigma^{ij} + 8\pi T_{\mu\nu} \dot{k}^\mu \dot{k}^\nu \]

Incorporates the constraint and evolution equations of GR, the Gauss–Codazzi–Ricci equations governing the embedding of \( S \) in the spatial hypersurface, and the demand that \( S \) remain at a constant coordinate location. These equations incorporate no assumption of quasi-equilibrium.

Terms that vanish because we demand \( S \) be a MOTS, remain a MOTS, or because we demand the horizon to be in equilibrium are in RED.
AH/Quasi-Equilibrium Boundary Conditions

\[ \theta = 0 \]

\[ 0 = D(\beta_\perp - \alpha), \]

\[ \dot{s}^i \nabla_i \alpha = -\frac{1}{\sqrt{2}} \left[ \dot{\theta} \left( \theta - \frac{1}{\sqrt{2}} K \right) + \dot{\sigma}_{ij} \sigma^{ij} \right] (\beta_\perp - \alpha) \]

\[ -\frac{1}{\sqrt{2}} \left[ \dot{\theta} \left( \frac{1}{2} \theta - \frac{1}{\sqrt{2}} K \right) + \dot{D} \right] (\beta_\perp + \alpha). \]

\[ \tilde{s}^k \nabla_k \ln \psi = -\frac{1}{4} (\tilde{h}^{ij} \tilde{\nabla}_i \tilde{s}_j - \psi^2 J) \]

\[ \beta^i = \alpha \psi^{-2} \tilde{s}^i + B^i_\parallel \]

\[ J \tilde{s}^i \tilde{\nabla}_i \alpha = -\psi^2 (J^2 - JK + \tilde{D}) \alpha \]

\[ \tilde{h}_{ij} \equiv \psi^4 \tilde{h}_{ij} \]

\[ s^i \equiv \psi^{-2} \tilde{s}^i \]

\[ B^i_\parallel s_i = 0 \]

\[ \tilde{D} \equiv \psi^{-4} [\tilde{h}^{ij} (\tilde{\nabla}_i - J_i)(\tilde{\nabla}_j - J_j) - \frac{1}{2} \tilde{R} + 2 \tilde{\nabla}^2 \ln \psi] \]

\[ [\tilde{\nabla} \& \tilde{R} \text{ are compatible with } \tilde{h}_{ij}] \]

The conditions of quasi-equilibrium yield boundary conditions for 3 of the 5 constrained variables \((\psi, \alpha, \beta_\perp)\). The remaining two conditions are contained in the definition of \(\beta^i_\parallel\). This freedom is necessary to prescribe the spin of the black hole.
Defining the Spin of the Black Hole

The spin parameters $\beta^i_i$ can be defined by demanding that the MOTS be a *Killing horizon*. The time vector associated with quasi-equilibrium in the corotating frame must be null, forming the null generators of the horizon.

$$k^\mu \propto (n^\mu + s^\mu) \implies k^\mu = \left[1, \alpha s^i - \beta^i\right]$$

*This vector $k^\mu$ is null for any choice of $\beta^i$.*

In the frame where a black hole is not spinning, the null time vector has components $t^\mu = [1, \vec{0}]$.

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**Corotating Holes**

Corotating holes are at rest in the corotating frame, where we must pose boundary conditions. So,

$$k^\mu = \left[1, \alpha s^i - \beta^i\right] = [1, \vec{0}]$$

Thus we find

$$\beta^i = \alpha s^i \implies \beta^i_i = 0$$

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**Irrotational Holes**

Irrotational holes are at rest in the inertial frame. With the time vectors in the inertial and corotating frames related by

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \bar{t}} + \Omega \frac{\partial}{\partial \phi}$$

$$k^\mu = \left[1, \alpha s^i - \beta^i\right] = \left[1, -\Omega \left(\frac{\partial}{\partial \phi}\right)^i\right]$$

Thus we find

$$\beta^i = \alpha s^i + \Omega \left(\frac{\partial}{\partial \phi}\right)^i \implies \beta^i_i = \Omega \left(\frac{\partial}{\partial \phi}\right)^i$$
Summary of QE Formalism

\[ \gamma_{ij} = \psi^4 \tilde{\gamma}_{ij} \quad K_{ij} = \psi^{-10} \tilde{A}_{ij} + \frac{1}{3} \gamma_{ij} K \quad \tilde{A}_{ij} = \frac{\psi^6}{2\alpha}(\tilde{\mathbb{L}}\beta)^{ij} \quad \partial_t \tilde{\gamma}_{ij} = 0 \]

\[ \tilde{\nabla}^2 \psi - \frac{1}{8} \psi \tilde{R} - \frac{1}{12} \psi^5 K^2 + \frac{1}{8} \psi^{-7} \tilde{A}_{ij} \tilde{A}^{ij} = 0 \]
\[ \tilde{\Delta}_{\mathbb{L}\beta}^i - (\tilde{\mathbb{L}}\beta)^{ij} \tilde{\nabla}_j \ln \alpha \psi^{-6} = \frac{4}{3} \alpha \tilde{\nabla}^i K \]
\[ \tilde{\nabla}^2(\alpha \psi) - (\alpha \psi) \left[ \frac{1}{8} \tilde{R} + \frac{5}{12} \psi^4 K^2 + \frac{7}{8} \psi^{-8} \tilde{A}_{ij} \tilde{A}^{ij} \right] = \psi^5 \beta^i \tilde{\nabla}_i K \quad \partial_t K = 0 \]

\[ \tilde{s}^k \tilde{\nabla}_k \ln \psi |_S = -\frac{1}{4}(\tilde{h}_{ij} \tilde{\nabla}_i \tilde{s}_j - \psi^2 J)|_S \quad \theta = 0 \]

\[ \beta^i |_S = \left\{ \begin{array}{ll}
\alpha \psi^{-2} \tilde{s}^i |_S & \text{corotation} \\
\alpha \psi^{-2} \tilde{s}^i |_S + \Omega \tilde{h}^i_j \left( \frac{\partial}{\partial \phi} \right)^j |_S & \text{irrotation} \\
\end{array} \right. \]

\[ J \tilde{s}^i \tilde{\nabla}_i \alpha |_S = -\psi^2(J^2 - JK + \tilde{D})\alpha |_S \quad \mathcal{L}_\zeta \theta = 0 \]

\[ \psi |_{r \to \infty} = 1 \quad \beta^i |_{r \to \infty} = \Omega \left( \frac{\partial}{\partial \phi} \right)^i \quad \alpha |_{r \to \infty} = 1 \]

The only remaining freedom in the system is the choice of the orbital angular velocity, the initial spatial and temporal gauge, and the initial dynamical ("wave") content found in \( \Omega, \tilde{\gamma}_{ij}, \text{and } K \).
The Orbital Angular Velocity

- For a given choice of $\tilde{\gamma}_{ij}$ and $K$, we are still left with a family of solutions parameterized by the orbital angular velocity $\Omega$.
- Except for the case of a single spinning black hole, it is not reasonable to expect more than one value of $\Omega$ to correspond to a system in quasi-equilibrium.

GGB[8, 9] have suggested a way to pick the quasi-equilibrium value of $\Omega$:

$\Omega$ is chosen as the value for which the ADM mass $E_{\text{ADM}}$ equals the Komar mass $M_K$.

\[ E_{\text{ADM}} = \frac{1}{16\pi} \int_\infty \gamma^{ij} \nabla_k (\mathcal{G}_i^k - \delta_i^k \mathcal{G}) d^2S_j \]

Acceptable definition of the mass for arbitrary spacetimes.

\[ M_K = \frac{1}{4\pi} \int_\infty \gamma^{ij} (\tilde{\nabla}_i \alpha - \beta^k K_{ik}) d^2S_j \]

Acceptable definition of the mass only for stationary spacetimes.

$\mathcal{G}_{ij} \equiv \gamma_{ij} - f_{ij}$
Do the AH/QE BCs Yield a Well Posed System?

Single Black Hole tests:

- $\tilde{\gamma}_{ij}$ and $K$ from Kerr-Schild:
  - AH/QE BCs seem ill-conditioned with slow/no nonlinear convergence.
  - Replacing the BC on either $\alpha$ or $\beta_\perp$ with the proper Dirichlet data yields good convergence.
  - Replacing the BC on either $\alpha$ or $\beta_\perp$ with the wrong Dirichlet data yields good convergence.
  - Solving with Dirichlet BC replacing one of the BCs yields a solution that:
    ★ obeys the full AH/QE BCs
    ★ has $\partial_t \psi = 0$
    *(if the outer boundary is at $\infty$)*

- $\tilde{\gamma}_{ij} = f_{ij}$ and $K = 1/r^2$ or 0
  - Solving with Dirichlet BC replacing one of the BCs yields a solution that:
    ★ obeys the full AH/QE BCs
    ★ has $\partial_t \psi = 0$
    *(if the outer boundary is at $\infty$)*
References


