Hands-on Gravitational Wave Astronomy:
Extraction of Astrophysical Information from Simulated Signals

Teachers Guide

Introduction

In this activity you will calculate several properties of a pair of orbiting neutron stars known as a binary system. You will base these calculations on four plots of simulated data, the type of data that gravitational wave detectors will output. The activity procedure resembles the methods that gravitational wave astronomers will use to uncover clues about the neutron star binary based on the gravitational waves that the binary emits.

The movie *Einstein's Messengers* describes LIGO's search for gravitational waves. Ground-based detectors such as LIGO hunt for signals from astrophysical events such as the inspiraling orbit of a pair of neutron stars. An inspiral occurs as the orbiting system emits energy through the release of gravitational waves. As the pair loses energy, their orbit shrinks and their orbital period decreases. Finally the shrinkage results in "coalescence," the collision or merger of the two bodies. When a binary system's orbital period decays to a small fraction of a second*, detectors such as LIGO could begin registering the gravitational waves that the system emits. At earlier times in the process, when the neutron stars are orbiting more slowly and at a greater distance from each other, a space-based detector such as the forthcoming LISA could sense the long period, low-frequency gravitational waves that arise far from coalescence. The simulated data that you will use in this activity represents contributions that would come from both LIGO and LISA.

Note: You may have difficulty imagining that two neutron stars, each as massive as the sun yet only a few dozen kilometers in diameter, can orbit each other in tenths of a second or less, corresponding to frequencies of tens of orbits per second or higher. Welcome to the bizarre world of astrophysics.

Part One: Examining the Data Plots

You will receive four data plots that show gravitational wave signals from a binary system of neutron stars. The wave patterns on the plots represent the changes in the sensor signals at a gravitational wave detector. Notice the smooth appearance of the curves. The jumble of instrument noise that would appear in "real" data has been removed -- these curves represent pure characterizations of the gravitational wave signals.** On each plot, the horizontal axis represents time and the vertical axis represents wave amplitude. The "h" denoted on the vertical scale represents strain, a common measurement for reporting the amplitudes of gravitational waves. The value of strain is a ratio of the change in the detector that gravitational waves cause compared to the size of the detector itself. Notice that the strains on the plots are very small, such as 1×10^{-22}. This tells you that the effects of gravitational waves are tiny. Since strain is a ratio of two lengths, it is dimensionless.

**Note: The companion activity "Searching for Gravitational Waves in Noisy Data" provides a comparison between the noisy data that comes from the detectors and the pure gravitational waveforms that lie buried in the noise.

Examine the four plots and complete the table below. Find the wave periods on the plots by using a ruler to measure the time intervals in seconds between adjacent wave crests. The reciprocal of the period is the number of waves per second, or wave frequency (in Hertz or Hz). Measure the amplitude of the waves by finding the strain value at the wave crests.

The Name of the Data Plot	Describe the plot's appearance	Measure the gravitational wave period (seconds)	Measure the gravitational wave frequency (Hz)	Measure the gravitational wave amplitude, h
Far from Coalescence (a million years until coalescence occurs)	Smooth cosine, unchanging amplitude and period	1000 s	0.001 Hz	1×10^{-22}
1.0 Second Before Coalescence	The amplitude has increased; the priods are shorter	0.0075 s	133 Hz	2.5×10^{-19}
0.1 Second Before Coalescence	You can see a change in amplitude from one side of the plot to the other	0.0030 s	333 Hz	5×10^{-19}
Last Second Before Coalescence	*How do the wave periods and amplitudes change on this plot?* **The periods shrink with the passage of time across the plot and the amplitude grows, slowly in the first fractions of a second then significantly at the end**			

You will use the values from this table for questions 3, 6, 8, 9, and 11

1.   How does the data change as the binary system moves towards coalescence?

Near coalescence the gravitational wave period shrinks and the amplitude grows. *Far from coalescence neither property measurably changes.*

2.   What is happening in the binary system that causes these changes in the gravitational wave data? (Review the introduction)

The binary system loses energy through the emission of gravitational waves. This causes a decay in the system's orbit. As the orbit decreases in size, the orbital period also decreases.

Part Two: Using Gravitational Wave Mathematics

In this section you will use astrophysics equations that relate the data plots to properties of the binary system. Derivations of the equations are not provided. Your focus will be to use some of the equations to calculate interesting features of the system.

On the "Last Second Before Coalescence" plot, you noticed that the periods of gravitational waves from the binary system decrease as the system approaches coalescence. This indicates that the orbital period of the system is decreasing as it gives off energy in the form of gravitational waves. You can calculate the orbital period at any time t by using

$$P_{orb}(t) = \left(P_0^{8/3} - \frac{8}{3}kt \right)^{3/8} \qquad (1)$$

where P_0 is the orbital period at time $t = 0$ and k is a constant given by

$$k \equiv \frac{96}{5c^5}(2\pi)^{8/3}(G\mathcal{M})^{5/3} \qquad (2)$$

In equation (2) c is the speed of light, G is Newton's gravitational constant and $\mathcal{M}$ is the "chirp mass" of the binary system, a quantity that is based on the masses of both bodies in the system. To calculate a chirp mass, use the expression

$$\mathcal{M} \equiv (M_1 M_2)^{3/5}(M_1 + M_2)^{-1/5} \qquad (3)$$

where M_1 and M_2 are the masses of the two bodies in the system. Note that the period used in equation (1) is the orbital period of the binary system, which is not necessarily the same as the period of gravitational waves that would register at a detector. Fortunately for a circularized binary system the two periods are simply related:

$$P_{orb}(t) = 2P_{gw}(t) \qquad (4)$$

Each orbit of the binary will produce two gravitational wave cycles. Use either $P_{orb}(t)$ or $P_{gw}(t)$ as called for in the following questions.

Orbital Separation and Distance to the Binary Using a Mass Approximation

The periods of gravitational waves remain very constant on the plot labeled "Far From Coalescence". This type of constant-period signal is named monochromatic radiation. Unfortunately a monochromatic signal mathematically limits our ability to calculate features of the system. We can circumvent this limitation by approximating mass values for the two neutron stars. Astronomical data suggest that neutron stars typically have masses equivalent to about 1.4 solar masses. In questions 3 – 8 you will use this mass

value for both stars to calculate the system's orbital separation and its distance from the earth.

3. Calculate the orbital separation of the pair of neutron stars when they are far from coalescence. Use Kepler's Third Law,

$$G(M_1 + M_2) = \left(\frac{2\pi}{P_{orb}}\right)^2 R^3 \qquad (5)$$

Use equation (4) to obtain the orbital period from the gravitational wave period (you already measured the gravitational wave period from the plot). Insert the approximated masses for the stars in kilograms. Then solve for R, the distance between the two masses. What percentage of the earth-moon separation (3.84 x 10^5 km) is the orbital separation of the neutron stars?

R = 3.4 x 10^8 m
Neutron star orbital separation = 88% of the earth-moon distance

4. Use the equation

$$v = \frac{2\pi}{P_{orb}} R \qquad (6)$$

to find v, the tangential velocity of the binary in meters per second. What would you say about the magnitude of this velocity?

V = 2.1 x 10^6 m/s, only about a factor of a hundred less than the speed of light!

5. To calculate the distance to the binary you will first find the chirp mass* for our system of two neutron stars, again with masses that are 1.4 times the mass of the sun. Use equation (3) to obtain the chirp mass.

Chirp mass $\mathcal{M}$ = 2.4 x 10^{30} kg

*Note: The term "chirp" comes from the decreasing period and increasing amplitude of the gravitational wave signal as the stars move towards coalescence. Shorter wave periods correspond to higher wave frequencies. Detectors such as LIGO can play their data stream through a set of speakers. A coalescence event would produce an up-swooping chirp sound in this audio stream as the stars whirled toward their merger. The likelihood of hearing a chirp in real time is very small since detector noise will most likely mask it. Careful data analysis will uncover the signal at a later time.

The expression for the gravitational waveform of a circular binary system is based on the cosine function. You may have noticed that the "Far From Coalescence" plot resembles a cosine plot. The equation that gives the gravitational waveform for a circular binary system is

$$h(t) = A(t)\cos(\phi(t)) \qquad (7)$$

where ϕ is the phase of the wave (0, $\pi/2$, π, etc.). If $A(t)$ were a constant, then the plot of $h(t)$ would be an unchanging cosine wave. A glance at the "Last Second Before Coalescence" plot reveals that this is definitely not the case. $A(t)$ itself changes with time and is called the "time-varying amplitude" of $h(t)$. The time-varying amplitude $A(t)$ is given by

$$A(t) = \frac{2(G\mathcal{M})^{5/3}}{c^4 r}\left(\frac{\pi}{P_{gw}(t)}\right)^{2/3} \qquad (8)$$

Here c is the speed of light and r is the distance to the binary system. $\mathcal{M}$ is the chirp mass.

6. Obtain a value for $A(t)$ by reading the amplitude from a wave on the "Far From Coalescence" plot. Once you have obtained $A(t)$, you will know each term in equation (7) except for r, the distance to the binary. Solve equation (7) to find a value of r in meters.

r = 2.5 x 10^{20} meters

7. What percentage of the distance to the binary is the distance to our nearest star Proxima Centauri (4.22 light years)? To the center of the galaxy (26,000 light years)? Your results will show that scientists could detect the binary from nearly anywhere in our galaxy.

(Prox Centauri) / Binary Distance = 0.0037%
Galactic Center / Binary Distance = 23%

8. For this question assume that P_0 in equation (1) is the wave period at the start of the "Far From Coalescence" plot. What amount of time would be required for the orbital period to change by 1% of this original value? You will need to find a value for k by using equation (2).

k = 5 x 10^{-6} ; time = 4.8 x 10^{11} s or about 15,000 years

Distance to the Binary with a Calculated Chirp Mass

Once the binary system's orbital period is rapidly decreasing as the stars approach coalescence, the changing orbital period provides a way to calculate the distance to the binary without guessing the masses. At this point the gravitational waves are no longer monochromatic. Now you will calculate a more rigorous chirp mass and binary distance.

9. Assign the period you measured from the plot "1.0 Second Before Coalescence" to the variable P_0 in equation (1). Assign the period you measured from the plot "0.1 Second Before Coalescence" to the variable $P_{orb}(t)$ in equation (1). The value for the time (t) in equation (1) is then 0.9 second. Now solve equation (1) for the corresponding value of k, the evolution constant.

Question #9: Evolution constant k = 5.2 x 10^{-6}

10. You now know all the values in equation (2) except the chirp mass. Calculate the chirp mass $\mathcal{M}$ by solving equation (2).

Chirp mass $\mathcal{M}$ = 2.5 x 10^{30} kg

11. Now re-calculate the distance to the binary using equation (8). Use your calculated chirp mass from question 10 and make sure that you use the appropriate values of P_{gw} and $A(t)$ from the plots. Solve equation (8) twice, once for 1.0 second before coalescence and another instance for 0.1 second before coalescence. Average your two distance values.

1 s = 2.7 x 10^{20} m
0.1 s = 2.5 x 10^{20} m
Average = 2.6 x 10^{20} m

12. What would be the chirp mass for a binary system with a 10 solar mass black hole and a 0.3 solar mass white dwarf? What would you say appears to be the effect of unequal masses in the binary?

Chirp mass $\mathcal{M}$ = 2.4 x 10^{30} kg

The effect of the black hole's large mass on the chirp mass is cancelled by the white dwarf. As observers, scientists would have difficulty distinguishing between the equal-mass and unequal-mass binaries in questions (11) and (12).

Conclusion

Gravitational wave astronomers use techniques similar to those in this activity. At this writing the world still waits for the first detection of gravitational waves, so no scientist has performed these calculations on actual gravitational wave data. Theoretical astrophysicists have developed these techniques using simulated data and mathematical modeling. The detection of gravitational waves will allow the theorists to compare their concepts, methods and models to experimental data.

Write a conclusion in which you describe what you have learned from your work in the activity. What are some specific ideas or skills that you have learned or improved upon? What information did you learn by doing the calculations? What insights did you gain about the process of analyzing gravitational wave data?