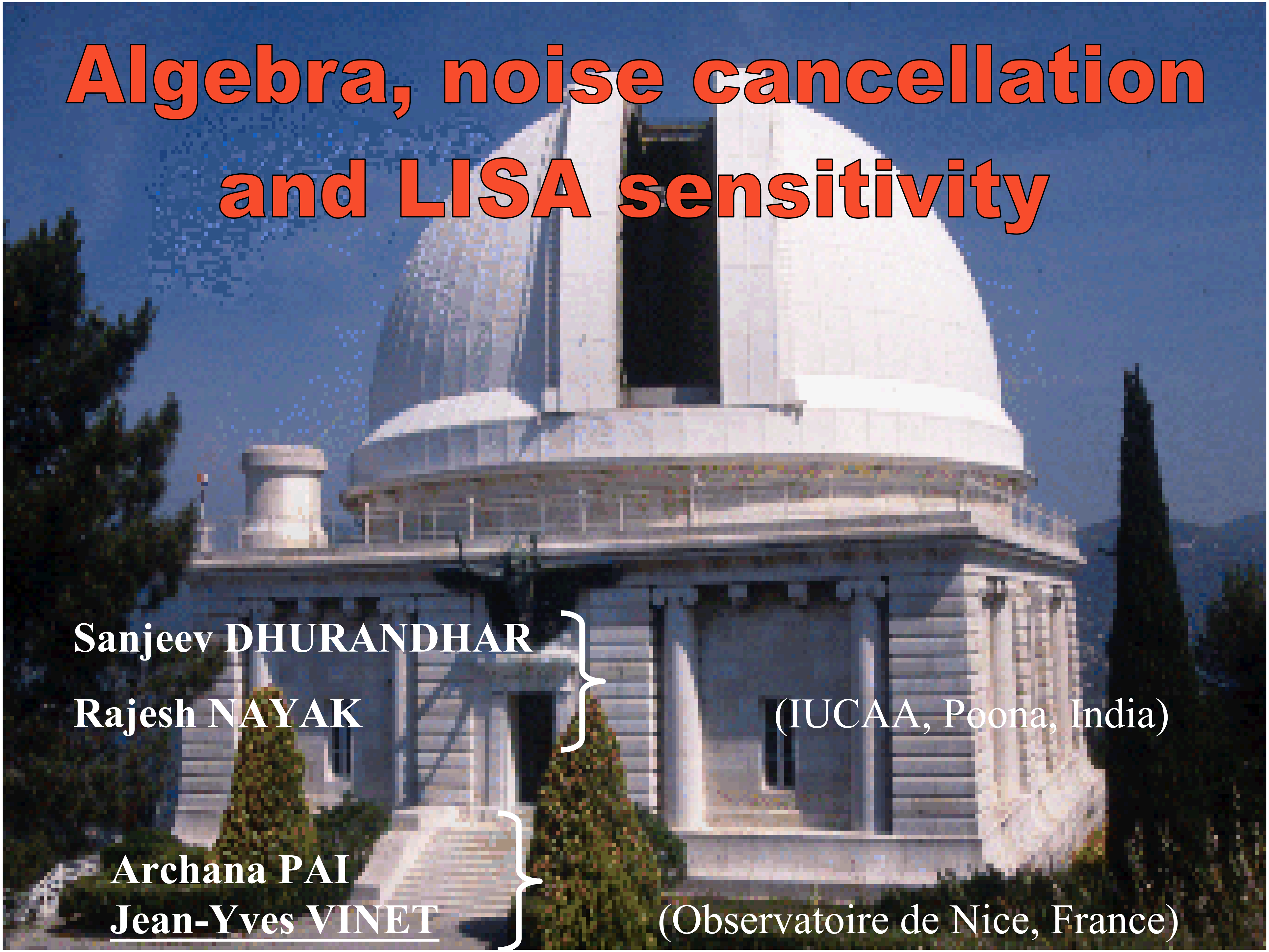

Algebra, noise cancellation and LISA sensitivity

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A **general** method for finding **all** combinations of data cancelling the laser noise and the bench position noise

Optimization of the combinations

A complement to the work of Armstrong, Estabrook & Tinto

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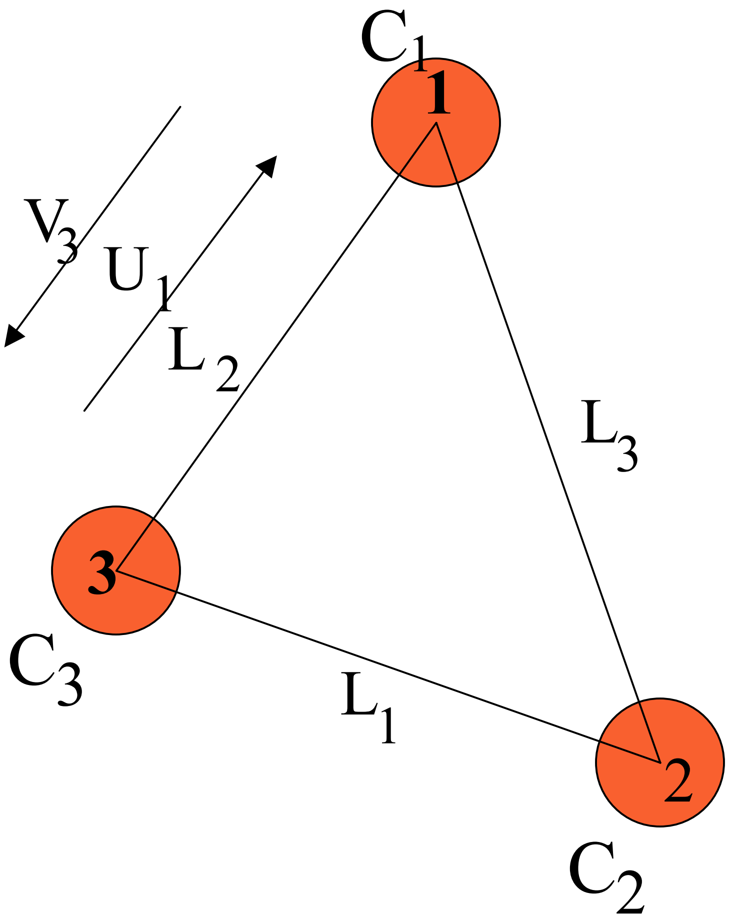


Lasers frequency noise ⇨ $C(t) = \dfrac{\delta\nu(t)}{\nu_0}$

Apparent Doppler data at each node :

$$U_1(t) = C_3(t - L_2) - C_1(t)$$

$$U_2(t) = C_1(t - L_3) - C_2(t)$$

$$U_3(t) = C_2(t - L_1) - C_3(t)$$

$$V_1(t) = C_1(t) - C_2(t - L_3)$$

$$V_2(t) = C_2(t) - C_3(t - L_1)$$

$$V_3(t) = C_3(t) - C_1(t - L_2)$$

Delay operators :

$$(x * f)(t) \equiv f(t - L_1)$$
$$(y * f)(t) \equiv f(t - L_2)$$
$$(z * f)(t) \equiv f(t - L_3)$$

Any combination of data with delays can be expressed as :

$$\Gamma(t) = \sum_{1}^{3} p_i(x, y, z) V_i(t) + \sum_{1}^{3} q_i(x, y, z) U_i(t)$$

Where the p_i, q_i are arbitrary *formal* polynomials

Data streams in matrix form :

$$\begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} = \begin{bmatrix} -1 & 0 & y \\ z & -1 & 0 \\ 0 & x & -1 \end{bmatrix} . \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix}$$

$$\begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} = \begin{bmatrix} 1 & -z & 0 \\ 0 & 1 & x \\ y & 0 & 1 \end{bmatrix} . \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix}$$

The U,V can be expressed as

$$\vec{U} = -\mu.\vec{C}, \qquad \vec{V} = \mu^T.\vec{C}$$

So that any combination is of the form :

$$\Gamma = (\vec{p}^T.\mu^T - \vec{q}^T.\mu).\vec{C} = (\mu.\vec{p} - \mu^T.\vec{q})^T.\vec{C}$$

Cancelling the laser frequency noise means :

$$\mu.\vec{p} - \mu^T.\vec{q} = 0$$

This is an algebraic condition on the combinations defined by $\vec{p}, \vec{q}$

It is not possible to arbitrarily take $\vec{q}$ and then

$$\vec{p} = (\mu^{-1}.\mu^{T}).\vec{q}$$

Because for expressing a time delay combination,

$\vec{p}$ must be a vector of polynomials in (x,y,z)

The equations are in detail :

$$p_1 - yp_3 - q_1 + zq_2 = 0$$
$$p_2 - zp_1 - q_2 + xq_3 = 0$$
$$p_3 - xp_2 - q_3 + yq_1 = 0$$

Gaussian elimination of p_1, p_2

$$p_1 = q_1 + yp_3 - zq_2, \qquad p_2 = q_2 + zp_1 - xq_3$$

Master equation

$$(1 - xyz)\,p_3 + (y - xz)q_1 + x(z^2 - 1)q_2 + (x^2 - 1)q_3 = 0$$

Let

$$f_1 \equiv 1 - xyz, f_2 = y - xz, f_3 = x(z^2 - 1), f_4 = x^2 - 1$$

Solutions of $\sum_1^4 \beta_i f_i = 0$ define the

module of syzygies, S of $\{f_1, f_2, f_3, f_4\}$

The set of all polynomials of the form

$$P(x, y, z) = \sum_1^4 \alpha_i(x, y, z) f_i(x, y, z)$$

where the α_i are arbitrary polynomials

Define an ***ideal*** on the ***ring*** of polynomials

The ideal admits a ***Gröbner basis*** $\{g_i\}$

from which generators of ***S*** can be found

One finds (standard softs),

$$g_1 = x - yz, \qquad g_2 = y^2 - 1, \qquad g_3 = z^2 - 1$$

From what a set of generators can be derived :

$$X^{(1)} = (xz - y, 0, z^2 - 1, 0, yz - x, z^2 - 1)$$

$$X^{(2)}(x, y, z, x, y, z) \Longleftrightarrow \zeta$$

$$X^{(3)}(1, z, xz, 1, xy, y) \Longleftrightarrow \alpha$$

$$X^{(4)} = (xy, 1, x, z, 1, yz) \Longleftrightarrow \beta$$

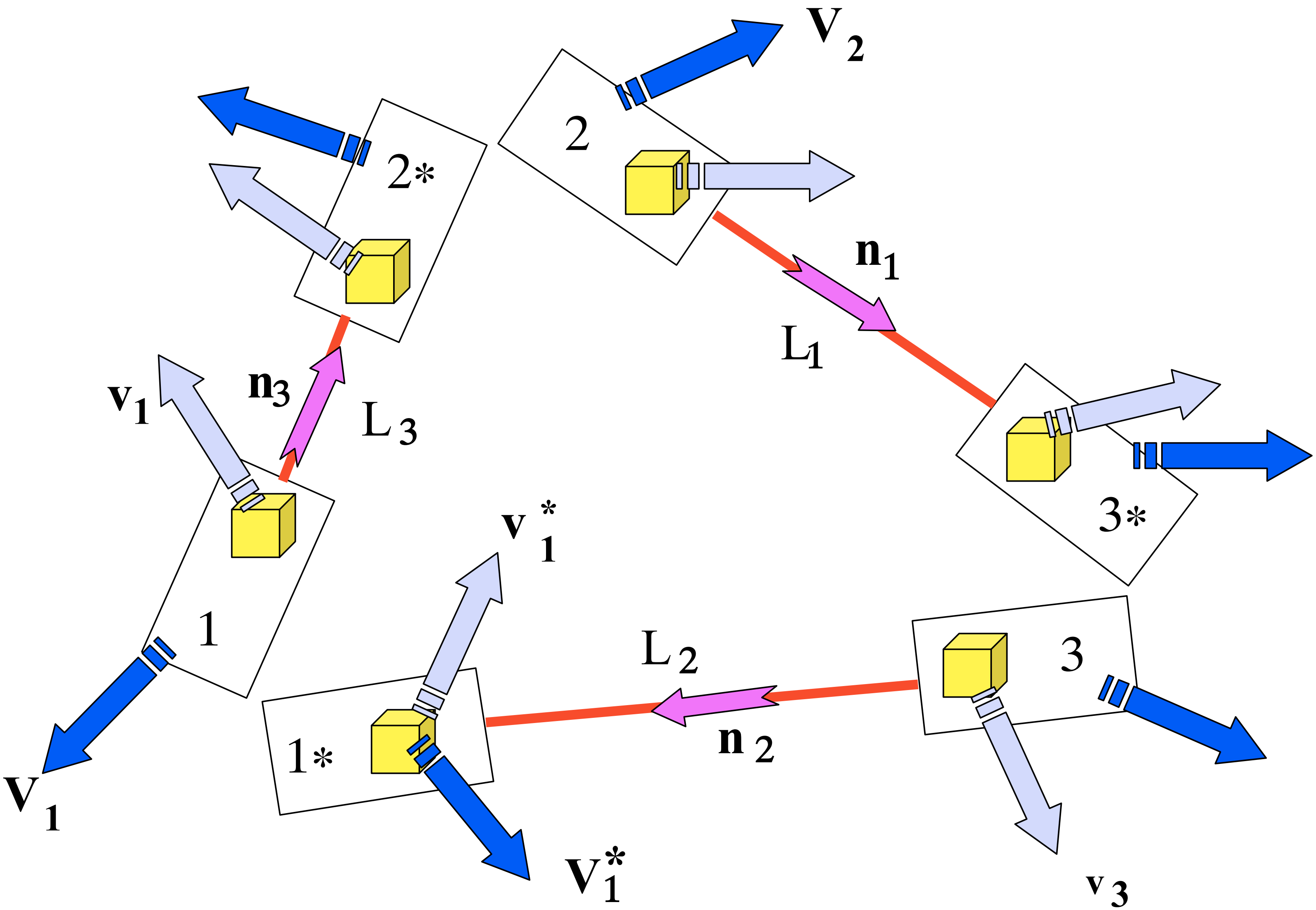
$\Updownarrow$ (between $X^{(1)}$ and $X^{(4)}$) **z**$\zeta-\gamma$

2 key points :

*These generators determine **all** laser-noise-cancelling time-delay data combinations*

*The method can be extended to **any number** of spacecrafts (LISA++ !)*

Bench noise cancellation



Bench noise cancellation

$$U_1 = C_3(t-L_2) - \vec{n}_2.\vec{V}_3(t-L_2) - C_1^*(t) - \vec{n}_2.\vec{V}_1^*(t) + 2\vec{n}_2.\vec{\mathrm{v}}_1^*(t)$$

$$V_1 = C_1(t) - \vec{n}_3.\vec{V}_1(t) - C_2^*(t-L_3) - \vec{n}_3.\vec{V}_2^*(t-L_3) + 2\vec{n}_3.\vec{\mathrm{v}}_1(t)$$

$$\left.\begin{aligned} z_1 &= C_1 - C_1^* + \eta_1 - 2\vec{n}_3.\vec{V}_1 + 2\vec{n}_3.\vec{\mathrm{v}}_1 \\ z_1^* &= C_1^* - C_1 + \eta_1 + 2\vec{n}_2.\vec{V}_1^* - 2\vec{n}_2.\mathrm{v}_1^* \end{aligned}\right\} \text{Inside spacecraft \#1}$$

One can define :

$$\widetilde{C}_1 \equiv C_1 - \vec{n}_3.\vec{V}_1, \qquad \widetilde{C}_1^* = C_1^* + \vec{n}_2.\vec{V}_1^*$$

Bench noise cancellation

$$U_1 = E_2 \tilde{C}_3 - \tilde{C}_1^* + 2\vec{n}_2.\vec{\mathrm{v}}_1^*, \qquad V_1 = \tilde{C}_1 - E_3 \tilde{C}_2^* + 2\vec{n}_3.\vec{\mathrm{v}}_1$$

$$Z_1 = \frac{1}{2}\left(z_1 - z_1^*\right) + \vec{n}_3.\vec{\mathrm{v}}_1 + \vec{n}_2.\vec{\mathrm{v}}_1^*$$

(+ circ. Perm.)

Combination suppressing the bench noises :

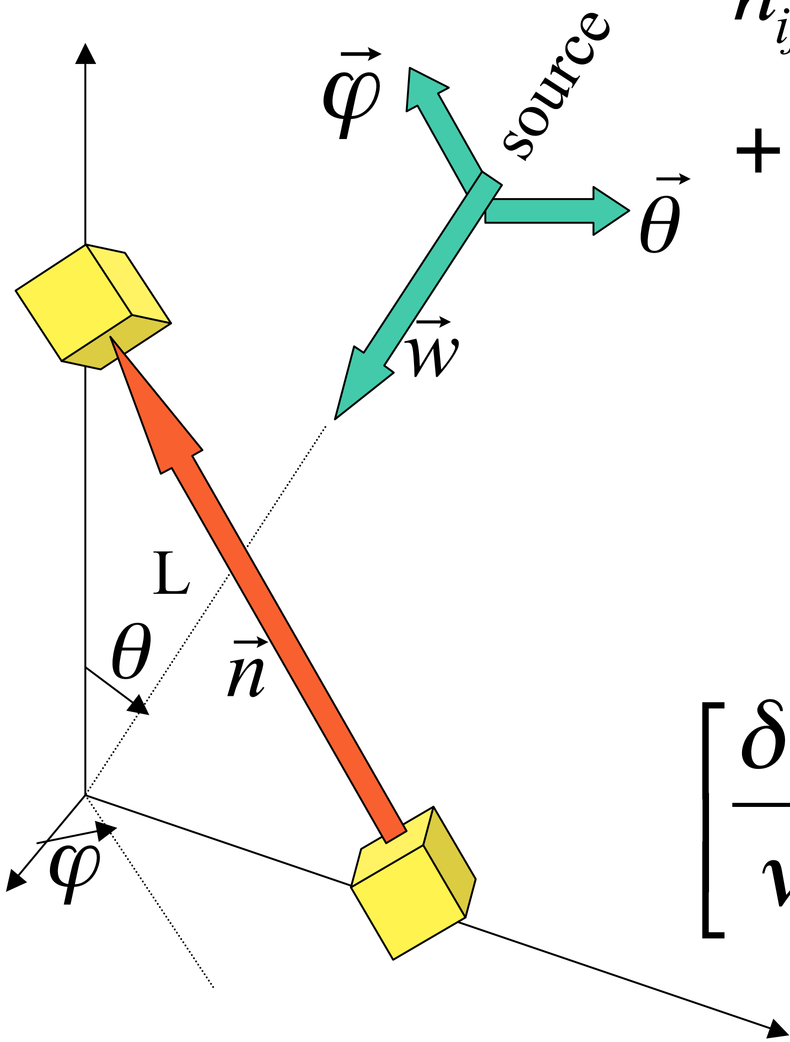
$$\sum_{n \le 3} q_n U_n + p_n V_n + r_n Z_n = 0$$

Because of the mapping $C \rightarrow \tilde{C}$

The generators of the Solutions are the same as For the laser noise cancellation

No suppression of the noise caused by the test-masses residual motion

One way gravitational response



$$h_{ij}(t) = h_+(t - \vec{w}.\vec{r})(\theta_i\theta_j - \varphi_i\varphi_j)$$
$$+ h_\times(t - \vec{w}.\vec{r})(\theta_i\varphi_j + \varphi_i\theta_j)$$

$$\xi_+ = (\vec{\theta}.\vec{n})^2 - (\vec{\varphi}.\vec{n})^2$$
$$\xi_\times = 2(\vec{\theta}.\vec{n})(\vec{\varphi}.\vec{n})$$

$$\left[\frac{\delta\nu}{\nu}\right]_{U,V} = F_{U,V+}h_+ + F_{U,V\times}h_\times$$

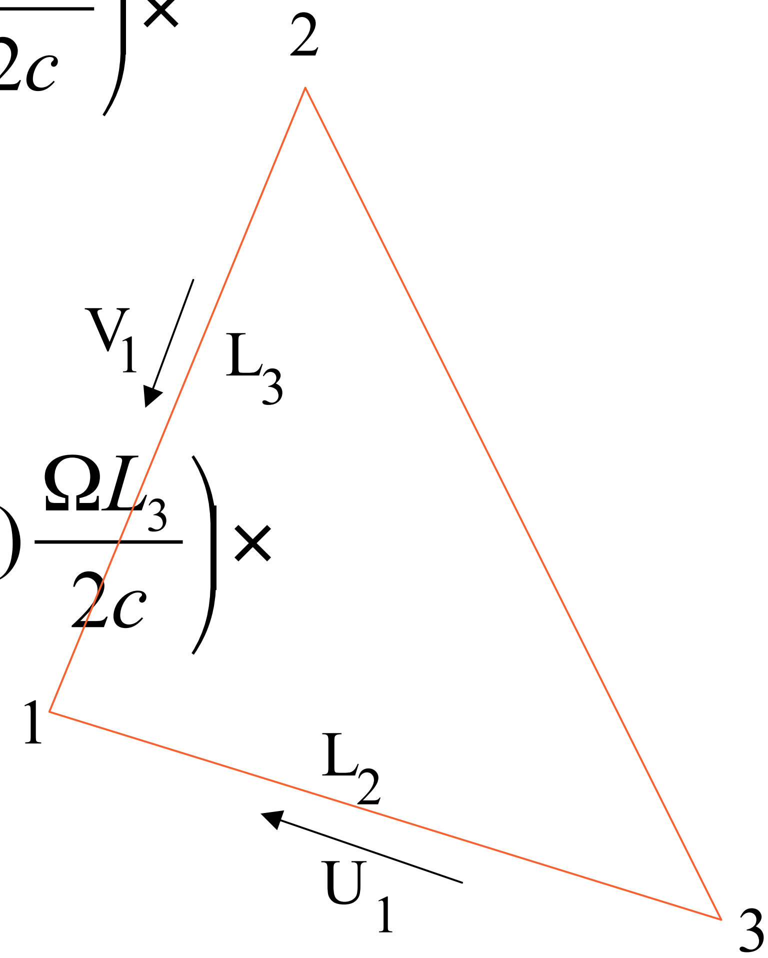
One-way transfer functions, Fourier domain $\Omega = 2\partial f_g$

$$F_{U_1+,\times} = \frac{\Omega L_2}{2c}\,\mathrm{sinc}\left(\frac{1}{2}(1-\vec{w}.\vec{n}_2)\frac{\Omega L_2}{2c}\right)\times$$

$$\exp\left[i\frac{\Omega L_2}{2c}\left(1-\vec{w}.\vec{r}_3/L_2\right)\right].\xi_{2+,\times}\Big|_{\Omega=2\pi f_g}$$

$$F_{V_1+,\times} = -\frac{\Omega L_3}{2c}\,\mathrm{sinc}\left(\frac{1}{2}(1+\vec{w}.\vec{n}_3)\frac{\Omega L_3}{2c}\right)\times$$

$$\exp\left[i\frac{\Omega L_3}{2c}\left(1-\vec{w}.\vec{r}_3/L_3\right)\right].\xi_{3+,\times}$$



Gravitational transfer function for the (i)th combination :

$$F^{(i)}{}_{+,\times} = \sum_{k} \left(p_k^{(i)} F_{V_k} \xi_{k''+,\times} + q_k^{(i)} F_{U_k} \xi_{k'+,\times} \right)$$

Noise for the same combination :

$$S^{(i)} = S^{(\mathrm{i})}_{\mathrm{proofmass}} + S^{(\mathrm{i})}_{\mathrm{shot\ noise}}$$

Structure of U and V

$$S^{(\mathrm{i})}_{\mathrm{proffmasses}} = \sum_{k=1}^{3} \left(\left| 2p_k^{(i)} + r_k^{(i)} \right|^2 + \left| 2q_k^{(i)} + r_k^{(i)} \right|^2 \right) S_{\mathrm{proofmass}}$$

$$S^{(\mathrm{i})}_{\mathrm{shotnoise}} = \sum_{k=1}^{3} \left(\left| p_k^{(i)} \right|^2 + \left| q_k^{(i)} \right|^2 \right) S_{\mathrm{shotnoise}}$$

Signal to noise ratio

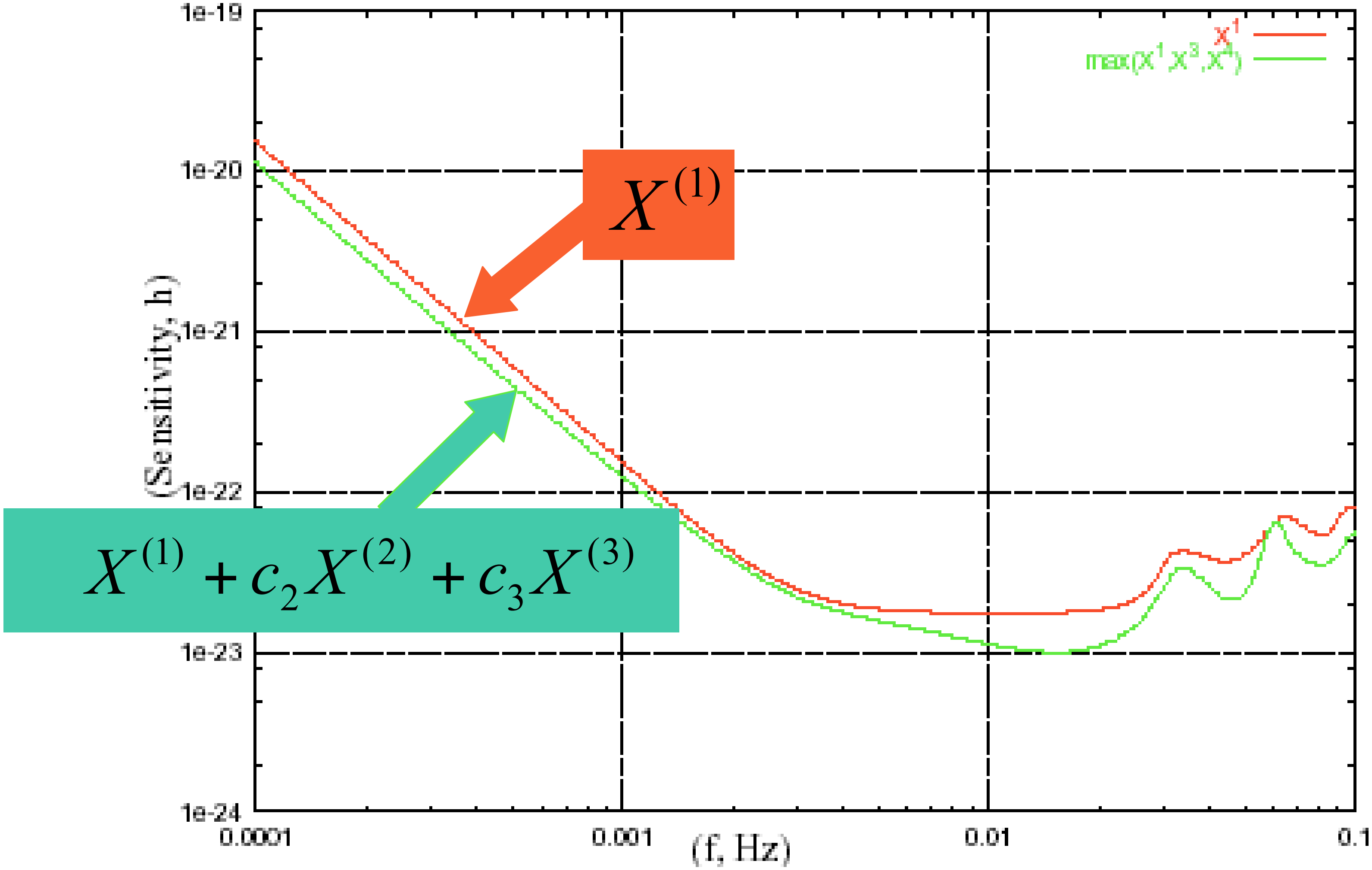
$$\mathrm{SNR}^{(i)} = \frac{\left| \mathrm{F}_{+}^{(i)} \mathrm{h}_{+} + \mathrm{F}_{\times} \mathrm{h}_{\times} \right|^2}{\mathrm{N}^{(i)}}$$

Maximization of the SNR :

$$Y = X^{(1)} + c_2 X^{(2)} + c_3 X^{(3)} + c_4 X^{(4)}$$

The c_i being optimal polynomials : we first look for real numbers…

Example of optimization



Optimization in the LISA plane