Generalized Harmonic Evolutions of Binary Black Hole Spacetimes

Lee Lindblom
Caltech

Collaborators: Duncan Brown, Larry Kidder, Geoffrey Lovelace, Robert Owen, Oliver Rinne, Harald Pfeiffer, Mark Scheel, Saul Teukolsky

The Pennsylvania State University
1 February 2007
Generalized Harmonic Evolutions of Binary Black Hole Spacetimes

Lee Lindblom
Caltech

Collaborators: Duncan Brown, Larry Kidder, Geoffrey Lovelace, Robert Owen, Oliver Rinne, Harald Pfeiffer, Mark Scheel, Saul Teukolsky

The Pennsylvania State University
1 February 2007

- Generalize Harmonic (GH) gauge conditions.
Generalized Harmonic Evolutions of Binary Black Hole Spacetimes

Lee Lindblom
Caltech

Collaborators: Duncan Brown, Larry Kidder, Geoffrey Lovelace, Robert Owen, Oliver Rinne, Harald Pfeiffer, Mark Scheel, Saul Teukolsky

The Pennsylvania State University
1 February 2007

- Generalize Harmonic (GH) gauge conditions.
- Constraint damping in the GH system.
Generalized Harmonic Evolutions
of Binary Black Hole Spacetimes

Lee Lindblom
Caltech

Collaborators: Duncan Brown, Larry Kidder, Geoffrey Lovelace,
Robert Owen, Oliver Rinne, Harald Pfeiffer,
Mark Scheel, Saul Teukolsky

The Pennsylvania State University
1 February 2007

- Generalize Harmonic (GH) gauge conditions.
- Constraint damping in the GH system.
- Moving Black Holes.
Generalized Harmonic Evolutions of Binary Black Hole Spacetimes

Lee Lindblom
Caltech

Collaborators: Duncan Brown, Larry Kidder, Geoffrey Lovelace, Robert Owen, Oliver Rinne, Harald Pfeiffer, Mark Scheel, Saul Teukolsky

The Pennsylvania State University
1 February 2007

- Generalize Harmonic (GH) gauge conditions.
- Constraint damping in the GH system.
- Moving Black Holes.
- Binary Black Hole Evolutions.
We often decompose the 4-metric into its 3+1 parts:

\[ ds^2 = \psi_{ab} dx^a dx^b = -N^2 dt^2 + g_{ij} (dx^i + N^i dt)(dx^j + N^j dt). \]

The lapse \( N \) and shift \( N^i \) specify how coordinates are laid out on a spacetime manifold:

\[ \vec{n} = \partial_{\tau} = (\partial_t - N^k \partial_k)/N. \]
Methods of Specifying Spacetime Coordinates

- We often decompose the 4-metric into its 3+1 parts:
  \[ ds^2 = \psi_{ab} dx^a dx^b = -N^2 dt^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^j dt). \]
  The lapse \( N \) and shift \( N^i \) specify how coordinates are laid out on a spacetime manifold: \( \vec{n} = \partial_\tau = (\partial_t - N^k \partial_k)/N. \)

- An alternate way to specify the coordinates is through the generalized harmonic gauge source function \( H^a \):

- Let \( H^a \) denote the function obtained by the action of the scalar wave operator on the coordinates \( x^a \):
  \[ H^a \equiv \nabla^c \nabla_c x^a = -\Gamma^a, \]
  where \( \Gamma^a = \psi^{bc} \Gamma^a_{bc} \) and \( \psi_{ab} \) is the 4-metric.
Methods of Specifying Spacetime Coordinates

- We often decompose the 4-metric into its 3+1 parts:
  \[ ds^2 = \psi_{ab}dx^adx^b = -N^2 dt^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^j dt). \]
  The lapse \( N \) and shift \( N^i \) specify how coordinates are laid out on a spacetime manifold: \( \vec{n} = \partial_\tau = (\partial_t - N^k \partial_k)/N. \)

- An alternate way to specify the coordinates is through the generalized harmonic gauge source function \( H^a: \)

- Let \( H^a \) denote the function obtained by the action of the scalar wave operator on the coordinates \( x^a: \)

\[ H^a \equiv \nabla^c \nabla_c x^a = -\Gamma^a, \]

where \( \Gamma^a = \psi^{bc}\Gamma^a_{bc} \) and \( \psi_{ab} \) is the 4-metric.

- Specifying coordinates by the generalized harmonic (GH) method can be accomplished by choosing a gauge-source function \( H_a(x, \psi) = \psi_{ab}H^b, \) and requiring that \( H_a(x, \psi) = -\Gamma_a = -\Gamma_{abc}\psi^{bc}. \)
Important Properties of the GH Method

The Einstein equations are manifestly hyperbolic when coordinates are specified using a GH gauge function:

$$R_{ab} = -\frac{1}{2} \psi^{cd} \partial_c \partial_d \psi_{ab} + \nabla(a \Gamma_b) + F_{ab}(\psi, \partial \psi),$$

where $\psi_{ab}$ is the 4-metric, and $\Gamma_a = \psi^{bc} \Gamma_{abc}$. The vacuum Einstein equation, $R_{ab} = 0$, has the same principal part as the scalar wave equation when $H_a(x, \psi) = -\Gamma_a$ is imposed.
Important Properties of the GH Method

- The Einstein equations are manifestly hyperbolic when coordinates are specified using a GH gauge function:

\[ R_{ab} = -\frac{1}{2} \psi^{cd} \partial_c \partial_d \psi_{ab} + \nabla (a \Gamma_b) + F_{ab}(\psi, \partial \psi), \]

where \( \psi_{ab} \) is the 4-metric, and \( \Gamma_a = \psi^{bc} \Gamma_{abc} \). The vacuum Einstein equation, \( R_{ab} = 0 \), has the same principal part as the scalar wave equation when \( H_a(x, \psi) = -\Gamma_a \) is imposed.

- Imposing coordinates using a GH gauge function profoundly changes the constraints. The GH constraint, \( C_a = 0 \), where \( C_a = H_a + \Gamma_a \),

depends only on first derivatives of the metric. The standard Hamiltonian and momentum constraints, \( M_a = 0 \), are determined by the derivatives of the gauge constraint \( C_a \):

\[ M_a \equiv G_{ab} n^b = \left[ \nabla (a C_b) - \frac{1}{2} \psi_{ab} \nabla^c C_c \right] n^b. \]
Pretorius (based on a suggestion from Gundlach, et al.) modified the GH system by adding terms proportional to the gauge constraints:

\[ 0 = R_{ab} - \nabla(aC_b) + \gamma_0 \left[ n(aC_b) - \frac{1}{2} \psi_{ab} n^c C_c \right], \]

where \( n^a \) is a unit timelike vector field. Since \( C_a = H_a + \Gamma_a \) depends only on first derivatives of the metric, these additional terms do not change the hyperbolic structure of the system.
Pretorius (based on a suggestion from Gundlach, et al.) modified the GH system by adding terms proportional to the gauge constraints:

\[ 0 = R_{ab} - \nabla(aC_b) + \gamma_0 \left[ n(aC_b) - \frac{1}{2} \psi_{ab} n^c C_c \right], \]

where \( n^a \) is a unit timelike vector field. Since \( C_a = H_a + \Gamma_a \) depends only on first derivatives of the metric, these additional terms do not change the hyperbolic structure of the system.

Evolution of the constraints \( C_a \) follow from the Bianchi identities:

\[ 0 = \nabla^c \nabla_c C_a - 2\gamma_0 \nabla^c \left[ n(cC_a) \right] + C^c \nabla(cC_a) - \frac{1}{2} \gamma_0 \ n^c C_c C_c. \]

This is a damped wave equation for \( C_a \), that drives all small short-wavelength constraint violations toward zero as the system evolves (for \( \gamma_0 > 0 \)).
Numerical Tests of the New GH System

- 3D numerical evolutions of static black-hole spacetimes illustrate the constraint damping properties of our GH evolution system.
- These evolutions are stable and convergent when $\gamma_0 = \gamma_2 = 1$. 

![Graphs showing stability and convergence of constraint damping properties for different values of $\gamma_0$ and $\gamma_2$.]
Moving Black Holes in a Spectral Code

- Spectral: Excision boundary is a smooth analytic surface.
Moving Black Holes in a Spectral Code

- Spectral: Excision boundary is a smooth analytic surface.
  - Cannot add/remove individual grid points.

![Diagram showing horizon and grid points]

Solution:
Choose coordinates that smoothly track the location of the black hole.
Moving Black Holes in a Spectral Code

- Spectral: Excision boundary is a smooth analytic surface.
  - Cannot add/remove individual grid points.
- Straightforward method: regrid when holes move too far.

Problems:
- Regridding/interpolation is expensive.
- Difficult to get smooth extrapolation at trailing edge of horizon.
- Causality trouble at leading edge of horizon.

Solution:
- Choose coordinates that smoothly track the location of the black hole.
Moving Black Holes in a Spectral Code

- Spectral: Excision boundary is a smooth analytic surface.
  - Cannot add/remove individual grid points.
- Straightforward method: regrid when holes move too far.

Solution:
Choose coordinates that smoothly track the location of the black hole.
Moving Black Holes in a Spectral Code

- Spectral: Excision boundary is a smooth analytic surface.
  - Cannot add/remove individual grid points.
- Straightforward method: regrid when holes move too far.

Problems:
- Regridding/interpolation is expensive.
- Difficult to get smooth extrapolation at trailing edge of horizon.
- Causality trouble at leading edge of horizon.

Solution:
- Choose coordinates that smoothly track the location of the black hole.
Moving Black Holes in a Spectral Code

- Spectral: Excision boundary is a smooth analytic surface.
  - Cannot add/remove individual grid points.
- Straightforward method: regrid when holes move too far.
- Problems:
  - Regridding/interpolation is expensive.

![Diagram of black hole horizon and grid](https://via.placeholder.com/150)
Moving Black Holes in a Spectral Code

- Spectral: Excision boundary is a smooth analytic surface.
  - Cannot add/remove individual grid points.
- Straightforward method: regrid when holes move too far.
- Problems:
  - Regridding/interpolation is expensive.
  - Difficult to get smooth extrapolation at trailing edge of horizon.
Moving Black Holes in a Spectral Code

- Spectral: Excision boundary is a smooth analytic surface.
  - Cannot add/remove individual grid points.
- Straightforward method: regrid when holes move too far.
- Problems:
  - Regridding/interpolation is expensive.
  - Difficult to get smooth extrapolation at trailing edge of horizon.
  - Causality trouble at leading edge of horizon.

![Diagram of horizon and black hole](attachment:horizon_black_hole.png)
Moving Black Holes in a Spectral Code

- Spectral: Excision boundary is a smooth analytic surface.
  - Cannot add/remove individual grid points.
- Straightforward method: regrid when holes move too far.
- Problems:
  - Regridding/interpolation is expensive.
  - Difficult to get smooth extrapolation at trailing edge of horizon.
  - Causality trouble at leading edge of horizon.

Solution:
Choose coordinates that smoothly track the location of the black hole.
Moving Black Holes in a Spectral Code

- Spectral: Excision boundary is a smooth analytic surface.
  - Cannot add/remove individual grid points.
- Straightforward method: regrid when holes move too far.
- Problems:
  - Regridding/interpolation is expensive.
  - Difficult to get smooth extrapolation at trailing edge of horizon.
  - Causality trouble at leading edge of horizon.

Solution:
Choose coordinates that smoothly track the location of the black hole.
Moving Black Holes in a Spectral Code

- Spectral: Excision boundary is a smooth analytic surface.
  - Cannot add/remove individual grid points.
- Straightforward method: regrid when holes move too far.
- Problems:
  - Regridding/interpolation is expensive.
  - Difficult to get smooth extrapolation at trailing edge of horizon.
  - Causality trouble at leading edge of horizon.
- Solution:
  Choose coordinates that smoothly track the location of the black hole.
Evolving Black Holes in Rotating Frames

- Coordinates that rotate with respect to the inertial frames at infinity are needed to track the horizons of orbiting black holes.
- Evolutions of Schwarzschild in rotating coordinates are unstable.

Evolutions shown use a computational domain that extends to $r = 1000M$.

Angular velocity needed to track the horizons of an equal mass binary at merger is about $\Omega \approx 0.2/M$.

Problem caused by asymptotic behavior of metric in rotating coordinates: $\psi_{tt} \sim \rho^2\Omega^2$, $\psi_{ti} \sim \rho\Omega$, $\psi_{ij} \sim 1$. 

$\Omega = 0.2/M$, $\Omega = 0.02/M$, $\Omega = 0.0002/M$
Dual-Coordinate-Frame Evolution Method

- Single-coordinate frame method uses the one set of coordinates, $x^{\bar{a}} = \{\bar{t}, \bar{x}^i\}$, to define field components, $u^{\bar{a}} = \{\psi_{\bar{a}\bar{b}}, \Pi_{\bar{a}\bar{b}}, \Phi_{\bar{i}\bar{a}\bar{b}}\}$, and the same coordinates to determine these components by solving Einstein’s equation for $u^{\bar{a}} = u^{\bar{a}}(x^{\bar{a}})$:

$$\partial_{\bar{t}} u^{\bar{a}} + A^{\bar{k}\bar{a}}_{\bar{\beta}} \partial_{\bar{k}} u^{\bar{\beta}} = F^{\bar{a}}.$$
Dual-Coordinate-Frame Evolution Method

- Single-coordinate frame method uses the one set of coordinates, $x^\bar{a} = \{\bar{t}, \bar{x}^i\}$, to define field components, $u^{\bar{a}} = \{\psi_{\bar{a}\bar{b}}, \Pi_{\bar{a}\bar{b}}, \Phi_{\bar{i}\bar{a}\bar{b}}\}$, and the same coordinates to determine these components by solving Einstein’s equation for $u^{\bar{a}} = u^{\bar{a}}(x^\bar{a})$:

$$\partial_{\bar{t}} u^{\bar{a}} + A^{\bar{k}\bar{a}}_{\bar{b}} \partial_{\bar{k}} u^{\bar{b}} = F^{\bar{a}}.$$ 

- Dual-coordinate frame method uses basis vectors of one coordinate system to define components of fields, and a second set of coordinates, $x^a = \{t, x^i\} = x^a(x^\bar{a})$, to represent these components as functions, $u^{\bar{a}} = u^{\bar{a}}(x^a)$. 

Lee Lindblom (Caltech)

Generalized Harmonic BBH Evolutions
Dual-Coordinate-Frame Evolution Method

- Single-coordinate frame method uses the one set of coordinates, \( \bar{\mathbf{x}}^a = \{ \bar{t}, \bar{x}^i \} \), to define field components, \( \bar{u}^\alpha = \{ \psi_{\bar{a}\bar{b}}, \Pi_{\bar{a}\bar{b}}, \Phi_{\bar{i}\bar{a}\bar{b}} \} \), and the same coordinates to determine these components by solving Einstein’s equation for \( \bar{u}^\alpha = \bar{u}^\alpha(\bar{x}^a) \):

\[
\partial_{\bar{t}}\bar{u}^\alpha + A^k_{\bar{\alpha} \bar{\beta}} \partial_k \bar{u}^\beta = F^\alpha.
\]

- Dual-coordinate frame method uses basis vectors of one coordinate system to define components of fields, and a second set of coordinates, \( x^a = \{ t, x^i \} = x^a(\bar{x}^a) \), to represent these components as functions, \( \bar{u}^\alpha = u^\alpha(x^a) \).

- These functions are determined by solving the transformed Einstein equation:

\[
\partial_{\bar{t}}u^\alpha + \left[ \frac{\partial x^i}{\partial \bar{t}} \delta^\alpha_{\bar{\beta}} + \frac{\partial x^i}{\partial x^k} A^k_{\bar{\alpha} \bar{\beta}} \right] \partial_i \bar{u}^\beta = F^\alpha.
\]
Testing Dual-Coordinate-Frame Evolutions

- Single-frame evolutions of Schwarzschild in rotating coordinates are unstable, while dual-frame evolutions are stable:

  **Dual Frame Evolution**

  **Single Frame Evolution**

  - Dual-frame evolution shown here uses a comoving frame with $\Omega = 0.2/M$ on a domain with outer radius $r = 1000M$. 
Horizon Tracking Coordinates

- Coordinates must be used that track the motions of the holes.
- For equal mass non-spinning binaries, the centers of the holes move only in the $z = 0$ orbital plane.
- The coordinate transformation from inertial coordinates, $(\bar{x}, \bar{y}, \bar{z})$, to co-moving coordinates $(x, y, z)$,

$$
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix} = e^{a(\bar{t})} \begin{pmatrix}
\cos \varphi(\bar{t}) & -\sin \varphi(\bar{t}) & 0 \\
\sin \varphi(\bar{t}) & \cos \varphi(\bar{t}) & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
\bar{x} \\
\bar{y} \\
\bar{z}
\end{pmatrix},
$$

with $t = \bar{t}$, is general enough to keep the holes fixed in co-moving coordinates for suitably chosen functions $a(\bar{t})$ and $\varphi(\bar{t})$.
- Since the motions of the holes are not known \textit{a priori}, the functions $a(\bar{t})$ and $\varphi(\bar{t})$ must be chosen dynamically and adaptively as the system evolves.
Measure the comoving centers of the holes: $x_c(t)$ and $y_c(t)$, or equivalently $Q^x(t) = [x_c(t) - x_c(0)]/x_c(0)$ and $Q^y(t) = y_c(t)/x_c(t)$.

Choose the map parameters $a(t)$ and $\varphi(t)$ to keep $Q^x(t)$ and $Q^y(t)$ small.
Measure the comoving centers of the holes: $x_c(t)$ and $y_c(t)$, or equivalently $Q^x(t) = [x_c(t) - x_c(0)]/x_c(0)$ and $Q^y(t) = y_c(t)/x_c(t)$.

Choose the map parameters $a(t)$ and $\varphi(t)$ to keep $Q^x(t)$ and $Q^y(t)$ small.

Changing the map parameters by the small amounts, $\delta a$ and $\delta \varphi$, results in associated small changes in $\delta Q^x$ and $\delta Q^y$:

$$\delta Q^x = -\delta a, \quad \delta Q^y = -\delta \varphi.$$
Measure the comoving centers of the holes: $x_c(t)$ and $y_c(t)$, or equivalently $Q^x(t) = [x_c(t) - x_c(0)]/x_c(0)$ and $Q^y(t) = y_c(t)/x_c(t)$.

Choose the map parameters $a(t)$ and $\varphi(t)$ to keep $Q^x(t)$ and $Q^y(t)$ small.

Changing the map parameters by the small amounts, $\delta a$ and $\delta \varphi$, results in associated small changes in $\delta Q^x$ and $\delta Q^y$:

$$\delta Q^x = -\delta a, \quad \delta Q^y = -\delta \varphi.$$ 

Measure the quantities $Q^y(t)$, $dQ^y(t)/dt$, $d^2Q^y(t)/dt^2$, and set

$$\frac{d^3 \varphi}{dt^3} = \lambda^3 Q^y + 3 \lambda^2 \frac{dQ^y}{dt} + 3 \lambda \frac{d^2Q^y}{dt^2} = -\frac{d^3 Q^y}{dt^3}.$$ 

The solutions to this “closed-loop” equation for $Q^y$ have the form $Q^y(t) = (At^2 + Bt + C)e^{-\lambda t}$, so $Q^y$ always decreases as $t \to \infty$. 
Horizon Tracking Coordinates III

- In practice the coordinate maps are adjusted only at a prescribed set of adjustment times \( t = t_i \).
- In the time interval \( t_i < t < t_{i+1} \) we set:

\[
\varphi(t) = \varphi_i + (t - t_i) \frac{d\varphi_i}{dt} + \frac{(t - t_i)^2}{2} \frac{d^2\varphi_i}{dt^2} + \frac{(t - t_i)^3}{2} \left( \frac{2}{2} \frac{d^2Q_i^y}{dt^2} + \lambda \frac{dQ_i^y}{dt} + \lambda^3 \frac{Q_i^y}{3} \right),
\]

where \( Q^x, Q^y \), and their derivatives are measured at \( t = t_i \), so these maps satisfy the closed loop equation at \( t = t_i \).
Horizon Tracking Coordinates III

- In practice the coordinate maps are adjusted only at a prescribed set of adjustment times $t = t_i$.
- In the time interval $t_i < t < t_{i+1}$ we set:

$$\varphi(t) = \varphi_i + (t - t_i) \frac{d\varphi_i}{dt} + \frac{(t - t_i)^2}{2} \frac{d^2\varphi_i}{dt^2} + \frac{(t - t_i)^3}{2} \left( \lambda \frac{d^2 Q^y_i}{dt^2} + \lambda^2 \frac{dQ^y_i}{dt} + \lambda^3 \frac{Q^y_i}{3} \right),$$

where $Q^x_i$, $Q^y_i$, and their derivatives are measured at $t = t_i$, so these maps satisfy the closed loop equation at $t = t_i$.

- **This works!** We are now able to evolve binary black holes using horizon tracking coordinates until just before merger.
Evolving Binary Black Hole Spacetimes

- We can now evolve binary black hole spacetimes with excellent accuracy and computational efficiency through many orbits.

Head-on Merger Movie

Lapse-$\psi_4$ Movie
Evolving Binary Black Hole Spacetimes II

- Gravitational waveform and frequency evolution for the equal mass non-spinning BBH evolution.
Evolving Binary Black Hole Spacetimes III

- Initial steps in convergence testing the 15 orbit evolution:
Reducing Orbital Eccentricity

- Astrophysical BBH systems are expected to have almost circular orbits by the time of merger.
- Commonly used “quasi-circular” initial data approximate the small radiation reaction driven radial velocities by setting them to zero.
- Our group (Pfeiffer and Lovelace) have constructed better BBH initial data with radial velocities chosen to reduce the orbital eccentricity.
Comparing Waveforms for Low Eccentricity Orbits

- Orbital eccentricity has little effect on gravitational waveforms.
- Overlap integrals between the low eccentricity orbit waveforms and the “quasi-circular” waveforms are greater than 0.99.
- Graphs compare low eccentricity waveforms (black) with “quasi-circular” waveforms (red).