Gravitational Recoil: Theory and Application

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The recoil problem has received a tremendous amount of attention recently.

- Bekenstein (1973)
- Fitchett (1983)
Brute force approach
Multipolar analysis of brute force results
Psuedo-analytic methods
(interlude: supermassive kicks)
Analytic methods, Monte Carlo simulations
Astrophysical applications
The complex Weyl scalar $\Psi_4$ is defined by

$$\Psi_4(t, r, \theta, \phi) \equiv C_{abcd} n^a(m^b)^* n^c(m^d)^* = -(\ddot{h}_+ - i\ddot{h}_\times),$$

with

$$\vec{n} \equiv (\hat{t} - \hat{r})/\sqrt{2}$$

$$\vec{m}^* \equiv (\hat{\theta} - i\hat{\phi})/\sqrt{2}$$

Then the linear momentum flux in the waves is

$$\frac{dP_i}{dt} = \frac{r^2}{16\pi} \int d\Omega \frac{x_i}{r} \left| \int_{-\infty}^{t} dt \Psi_4 \right|^2$$
As in E&M, the gravitational field can be decomposed into a multipolar expansion

\[ Mr\Psi_4(t, \vec{r}) = \sum_{\ell m} -\ell - 2 C_{\ell m}(t) - 2 Y_{\ell m}(\theta, \varphi), \]

or in terms of mass- and current-multipoles,

\[ (\ell + 2) I^{\ell m} = -\frac{1}{\sqrt{2}} \left[ -2 C_{\ell m} + (-1)^m -2 C^*_{\ell - m} \right], \]

\[ (\ell + 2) S^{\ell m} = -\frac{i}{\sqrt{2}} \left[ -2 C_{\ell m} - (-1)^m -2 C^*_{\ell - m} \right]. \]

Schnittman et al. (2007), arXiv:0707.0301
Momentum flux from multipolar expansion

\[ F_z = \frac{1}{336\pi} \left[ 4\sqrt{14} \Re(i^{31} l^{21*}) - 14\Im(i^{21} S^{21*}) + 2\sqrt{35} \Re(l^{22} l^{32*}) \\
-28\Im(l^{22} S^{22*}) + 3\sqrt{7} l^{20} l^{30} + 3\sqrt{7} S^{20} S^{30} \\
+4\sqrt{14} \Re(S^{21} S^{31*}) + 2\sqrt{35} \Re(S^{22} S^{32*}) - 7\Im(l^{31} S^{31*}) \\
-14\Im(l^{32} S^{32*}) - 21\Im(l^{33} S^{33*}) + 2\sqrt{21} l^{30} l^{40} \\
+3\sqrt{35} \Re(l^{31} l^{41*}) + 6\sqrt{7} \Re(l^{32} l^{42*}) + 7\sqrt{3} \Re(l^{33} l^{43*}) \right]. \]

\[ F_x + iF_y = \frac{1}{672\pi} \left[ -28i S^{21} l^{22*} + 2\sqrt{14} l^{31} l^{22*} - 2\sqrt{210} l^{22} l^{33*} \\
+14i\sqrt{6} l^{20} S^{21*} - 14i\sqrt{6} S^{20} l^{21*} + 28i l^{21} S^{22*} + 2\sqrt{42} l^{30} l^{21*} \\
-4\sqrt{21} l^{20} l^{31*} - 4\sqrt{35} l^{21} l^{32*} - 7i\sqrt{6} S^{32} l^{33*} - 14\sqrt{6} l^{33} l^{44*} \\
+4\sqrt{21} S^{20} S^{31*} - 4\sqrt{35} S^{21} S^{32*} - 2\sqrt{210} S^{22} S^{33*} + 2\sqrt{42} S^{30} S^{21*} \\
+14i\sqrt{3} l^{30} S^{31*} - 14i\sqrt{3} S^{30} l^{31*} + 7i\sqrt{10} l^{31} S^{32*} - 7i\sqrt{10} S^{31} l^{32*} \\
-2\sqrt{105} l^{30} l^{41*} + 6\sqrt{7} l^{40} l^{31*} - 3\sqrt{70} l^{31} l^{42*} + 3\sqrt{14} l^{41} l^{32*} \\
+21\sqrt{2} l^{32} l^{43*} + 2\sqrt{14} S^{31} S^{22*} + 4\sqrt{2} l^{32} l^{33*} + 7i\sqrt{6} l^{32} S^{33*} \right], \]
Non-spinning, quasi-circular orbits

\[
\begin{align*}
F_z &= 0, \\
F_x + iF_y &= \frac{1}{672\pi} \left[ -28iS^{21}l^{22\ast} + 2\sqrt{14}l^{31}l^{22\ast} - 2\sqrt{210}l^{22}l^{33\ast} \\
&\quad + 14i\sqrt{6}l^{20}S^{21\ast} - 14i\sqrt{6}S^{20}l^{21\ast} + 28il^{21}S^{22\ast} + 2\sqrt{42}l^{30}l^{21\ast} \\
&\quad - 4\sqrt{21}l^{20}l^{31\ast} - 4\sqrt{35}l^{21}l^{32\ast} - 7i\sqrt{6}S^{32}l^{33\ast} - 14\sqrt{6}l^{33}l^{44\ast} \\
&\quad + 4\sqrt{21}S^{20}S^{31\ast} - 4\sqrt{35}S^{21}S^{32\ast} - 2\sqrt{210}S^{22}l^{33\ast} + 2\sqrt{42}S^{30}S^{21\ast} \\
&\quad + 14i\sqrt{3}l^{30}S^{31\ast} - 14i\sqrt{3}S^{30}l^{31\ast} + 7i\sqrt{10}l^{31}S^{32\ast} - 7i\sqrt{10}S^{31}l^{32\ast} \\
&\quad - 2\sqrt{105}l^{30}l^{41\ast} + 6\sqrt{7}l^{40}l^{31\ast} - 3\sqrt{70}l^{31}l^{42\ast} + 3\sqrt{14}l^{41}l^{32\ast} \\
&\quad + 21\sqrt{2}l^{32}l^{43\ast} + 2\sqrt{14}S^{31}S^{22\ast} + \sqrt{42}l^{42}l^{33\ast} + 7i\sqrt{6}l^{32}S^{33\ast} \right] \\
&\approx \frac{1}{672\pi} \left[ -28iS^{21}l^{22\ast} - 2\sqrt{210}l^{22}l^{33\ast} - 14\sqrt{6}l^{33}l^{44\ast} \right]
\end{align*}
\]

cf. Damour & Gopakumar (2006)
Recoil is dominated by modes with $\ell \leq 4$
Leading-order multipole modes for circular orbits

\[
S^{21} = -4i \sqrt{\frac{2\pi}{5}} \eta R \omega^3 e^{-i\phi} \left[ \frac{2}{3} \delta m R^2 \omega + \Delta^z \right],
\]

\[
I^{22} = 16i \sqrt{\frac{2\pi}{5}} \eta R^2 \omega^3 e^{-2i\phi},
\]

\[
I^{33} = 54 \sqrt{\frac{\pi}{21}} \eta \delta m R^3 \omega^4 e^{-3i\phi},
\]

\[
I^{44} = -\frac{256}{9} i \sqrt{\frac{2\pi}{7}} \eta (1 - 3\eta) R^4 \omega^5 e^{-4i\phi}.
\]
The ringdown flux is well-described by a collection of quasi-normal modes with different frequencies

\[ F_{\text{RD}}^{\ell m, \ell' m'} \approx F_{\text{match}}^{\ell m, \ell' m'} \exp[-i(\sigma_{\ell m 0} - \sigma^*_{\ell' m' 0})(t - t_{\text{match}})] \]

\[ \sigma_{\ell m 0} \equiv \omega_{\ell m 0} + i/\tau_{\ell m 0} \]

\[ |\omega_{210} - \omega_{220}| = 0.07/M \]

\[ |\omega_{220} - \omega_{330}| = 0.31/M \]

\[ |\omega_{330} - \omega_{440}| = 0.29/M \]

\[ \tau_{\ell m 0} \approx 12M \]
Ringdown phase is responsible for “anti-kick”
Equal-mass spinning BHs have almost no anti-kick
Why do planar spins give such large kicks?

- For equal-mass BHs with spins parallel to orbital axis, \( v_{\text{max}} \approx 500 \text{ km/s} \) Herrmann et al. (2007)
- When spins are in orbital plane, \( v_{\text{max}} \approx 4000 \text{ km/s} \) Campanelli et al. (2007)
- Three different effects explain this factor of \( 8 = 2 \times 2 \times 2 \)
  - Spin-orbit terms in weak-field limit
  - No rotation of flux during ringdown
  - Anomalous mode amplification during merger
Leading-order spin-orbit terms

\[ F_{SO} = \frac{16}{15} \eta^2 \omega^2 \frac{\Delta}{R^3} \left[ \hat{n} \times \Delta + (\hat{n} \times \hat{v})(\hat{v} \cdot \Delta) \right], \]
\[ \sim \Delta^z \quad (\text{spins along orbital axis}) \]
\[ \sim 2\Delta^p \sin \varphi \quad (\text{spins in orbital plane}) \]

Kidder (1995)

\[ F_x + iF_y = \frac{1}{336\pi} \left[ -14iS^{21}l^{22\ast} + \cdots \right] \]
\[ F_z = \frac{1}{336\pi} \left[ -28S(l^{22}S^{22\ast}) \right] + \cdots \]

Schnittman et al. (2007)
Flux rotation during ringdown

\[ F_{RD}^{\ell m, \ell' m'} \approx F_{\text{match}}^{\ell m, \ell' m'} \exp[-i(\sigma_{\ell m 0} - \sigma_{\ell' m' 0}^*)(t - t_{\text{match}})] \]
Anomalous amplification of $S^{22}$ mode

\[
S_{SO}^{21} = -4i \sqrt{\frac{2\pi}{5}} \eta R \omega^3 e^{-i\phi}(\Delta^z)
\]

\[
S_{SO}^{22} = -4i \sqrt{\frac{2\pi}{5}} \eta R \omega^3 e^{-i\phi}(-\Delta^x + i\Delta^y)
\]
Effective-one-body model

- EOB is re-summed formulation of conservative PN equations of motion \cite{Buonanno:1999bh}
- Smoothly match inspiral to linear combination of quasi-normal ringdown modes
- Vary point of matching around light ring

\begin{align*}
|v| \text{ (km/s)} & \begin{array}{c}
\text{NR} \\
\text{EOB } r_{\text{match}} = 3.0M \\
\text{EOB } r_{\text{match}} = 2.8M \\
\text{EOB } r_{\text{match}} = 2.75M \\
\text{EOB } r_{\text{match}} = 2.7M
\end{array} \\
(t-t_{\text{peak}})/M & \begin{array}{c}
0 \\
50 \\
100 \\
150 \\
200
\end{array}
\end{align*}
Monte Carlo simulations give distribution of recoil velocities for a wide range of spins and mass ratios.

\[ P(\nu_{\text{kick}} < \nu_{90}) = 90\% \]

\[ \eta = \frac{m_1 m_2}{m^2} \]

Schnittman & Buonanno (2007)
Very large kicks are relatively rare

- For $1 < m_1/m_2 < 10$, $a/M = 0.9$, and uniform spin orientation, the fraction of kicks greater than 500 km/s is 12% and only 3% have kicks greater than 1000 km/s.

- For “major mergers” with $m_1/m_2 < 4$, the typical kicks are larger, with $f_{500} = 0.31$ and $f_{1000} = 0.08$.

- In gas-poor (“dry”) mergers, the spin distribution is expected to be roughly uniform Schnittman (2004).

- In “wet” mergers, the torque from a circumbinary accretion disk should align the BH spins with the orbital axis, giving $v_{\text{max}} \approx 200$ km/s Bogdanovic et al.(2007).

- A Sloan search for broad AGN emission lines shifted relative to the host galaxy finds $f_{500} \leq 0.04$ and $f_{1000} \leq 0.0035$ for a sample of $> 2500$ objects Bonning et al.(2007).
In the beginning, there were many small galaxies.

These galaxies merged to form larger galaxies.

Most galaxies appear to host supermassive black holes.

If two galaxies merge, their central BHs will also merge.

$M_{\text{BH}} \sim \sigma^{4-5}$ and $v_{\text{escape}} \sim \sigma$

Valluri et al. (2005)
Simplified binary tree merger model

For each merger, the probability of ejection is $p_{ej}$

$p_{ej} = 0.1$

Schnittman (2007)
In each generation, some BHs get ejected

BH occupation fraction of $i^{th}$ generation: $f_i$

\[
f_{i+1} = 0 \times (1 - f_i)^2 + f_i(1 - f_i) + (1 - f_i)f_i + (1 - p_{ej})f_i^2 = f_i(2 - f_i - p_{ej}f_i)
\]

Taking the convergence limit of $f_{i+1} = f_i$ for large $i$, we find

\[
f_\infty = \frac{1}{1 + p_{ej}}.
\]
Even with large kicks, many BHs are retained

In limit of $p_{ej} \rightarrow 1$, $f_\infty \rightarrow 0.5$

$p_{ej} = 0.9$
More sophisticated merger model shows same basic result

- Initial mass function
  \[ \frac{dN}{dM} \sim M^{-1} \]
- \( M_{\text{BH}} \sim M_{\text{bulge}} \sim \sigma_{\text{bulge}}^{4.7} \)
- \( v_{\text{escape}} = 2\sigma_{\text{bulge}} \)
- Kick distribution from MC simulations with \( a/M = 0.9 \)
- Initial occupation fraction either 100\% or 40\%
- All solutions converge to \( f_{\infty} > 0.5 \)

\[ f_{\infty} \]

\[ f_{\text{BH}} \]

\[ f_{0} \]

\[ f_{1} \]

\[ f_{2} \]

\[ f_{3} \]

\[ f_{4} \]

\[ f_{\text{inf}} \]

Summary/Conclusions

- Full 3-D NR simulations serve as “answer key” for developing fast analytic models of binary BH inspiral, merger, and ringdown.
- Gravitational recoil is an ideal problem for testing these models, which must be faithful and accurate.
- Multipolar analysis helps identify the major issues and focus our efforts: where do analytic methods work well, and where do they fail?
- With improving confidence in our models, we will be able to make concrete predictions for the distribution of kicks in SMBH mergers, and test theory against observation.